Hidden Markov Models

This presentation was prepared based on material and slides from the books:

- S. Theodorids and K. Koutroumbas, "*Pattern Recognition, 4th Edition*", Academic Press, 2008
- S. Theodoridis, A. Pikrakis, K. Koutroumbas and D. Cavouras, "*Introduction to Pattern Recognition: a Matlab Approach*", Academic Press, 2010

CONTEXT DEPENDENT CLASSIFICATION

- Remember: Bayes rule

$$P(\omega_i|\underline{x}) > P(\omega_j|\underline{x}), \ \forall j \neq i$$

- Here: The class to which a feature vector belongs depends on:
  - Its own value
  - The values of the other features
  - An existing relation among the various classes

- This interrelation demands the classification to be performed simultaneously for all available feature vectors

- Thus, we will assume that the training vectors $\underline{x}_1, \underline{x}_2, \ldots, \underline{x}_N$ occur in sequence, one after the other and we will refer to them as **observations**

❖ The Context Dependent Bayesian Classifier

- ➢ Let $X : \{\underline{x}_1, \underline{x}_2, \ldots, \underline{x}_N\}$
- ➢ Let $\omega_i, \; i = 1,2,\ldots,M$
- ➢ Let Ω_i be a sequence of classes, that is

  $\Omega_i : \omega_{i1} \; \omega_{i2} \; \ldots \omega_{iN}$

  There are M^N of those

- ➢ Thus, the Bayesian rule can equivalently be stated as

$$\boxed{X \rightarrow \Omega_i : \;\; P(\Omega_i | X) > P(\Omega_j | X) \;\; \forall i \neq j, \;\; i, j = 1,2,\ldots, M^N}$$

❖ Markov Chain Models (for class dependence)

$$P(\omega_{i_k} | \omega_{i_{k-1}}, \omega_{i_{k-2}}, \ldots, \omega_{i_1}) = P(\omega_{i_k} | \omega_{i_{k-1}})$$

- NOW remember:

$$P(\Omega_i) = P(\omega_{i_1}, \omega_{i_2}, \ldots, \omega_{i_N}) =$$

$$= P(\omega_{i_N} \big| \omega_{i_{N-1}}, \ldots, \omega_{i_1}).$$

$$P(\omega_{i_{N-1}} \big| \omega_{i_{N-2}}, \ldots, \omega_{i_1}) \ldots P(\omega_{i_1})$$

or

$$\boxed{P(\Omega_i) = (\prod_{k=2}^{N} P(\omega_{i_k} \big| \omega_{i_{k-1}}))P(\omega_{i_1})}$$

- Assume:
  - $\underline{x}_i$ statistically mutually independent
  - The pdf in one class independent of the others, then

$$\boxed{p(X \big| \Omega_i) = \prod_{k=1}^{N} p(\underline{x}_k \big| \omega_{i_k})}$$

- From the above, the Bayes rule is readily seen to be equivalent to:

$$P(\Omega_i|X)(><)P(\Omega_j|X)$$

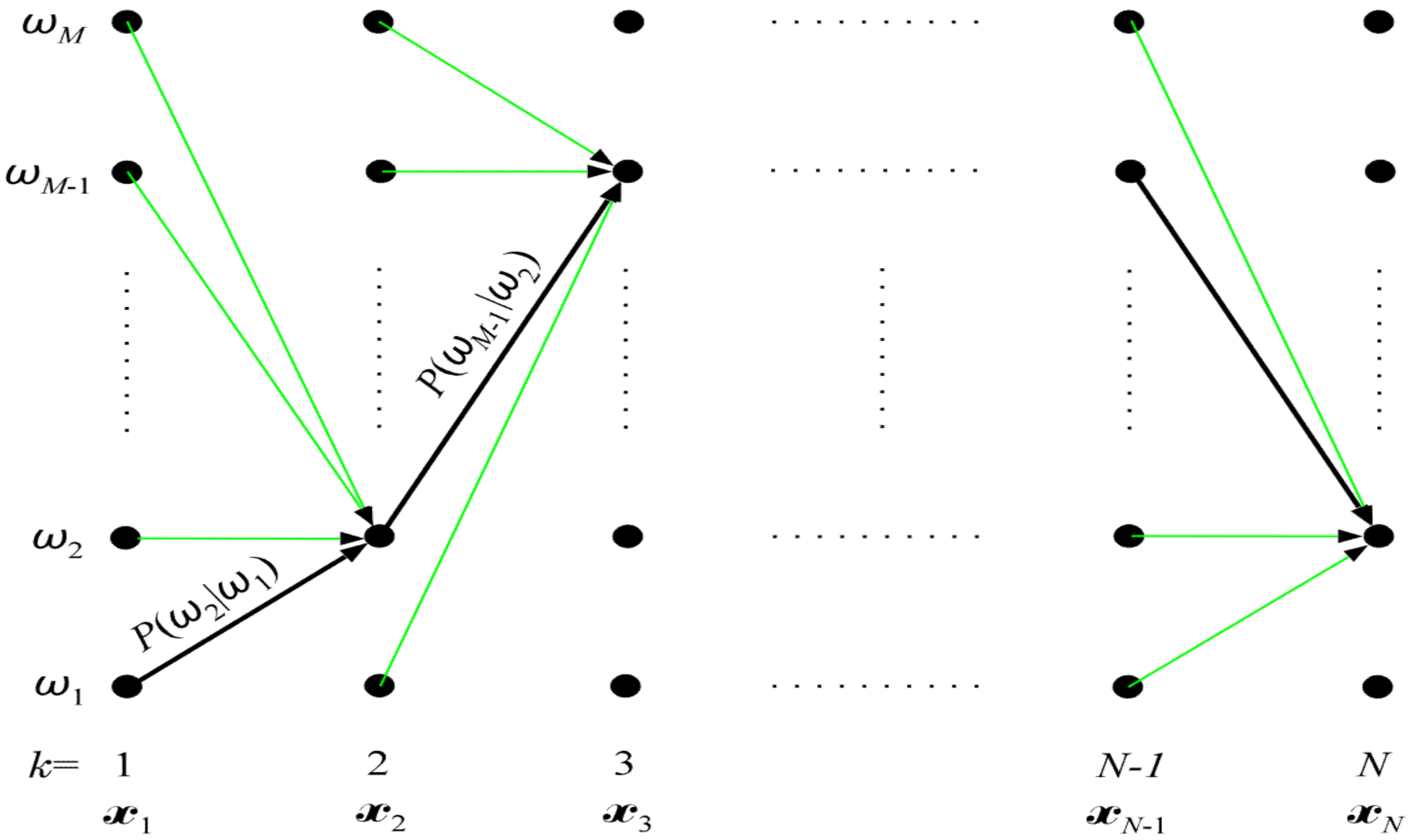
$$P(\Omega_i)p(X|\Omega_i)(><)P(\Omega_j)p(X|\Omega_j)$$

that is, it rests on

$$\boxed{\begin{array}{l} p(X|\Omega_i)P(\Omega_i) = P(\omega_{i_1})p(\underline{x}_1|\omega_{i_1}). \\ \displaystyle\prod_{k=2}^{N} P(\omega_{i_k}|\omega_{i_{k-1}})p(\underline{x}_k|\omega_{i_k}) \end{array}}$$

- To find the above maximum in brute-force task we need $O(NM^N)$ operations!!

❖ The Viterbi Algorithm



- Thus, each Ω_i corresponds to one path through the trellis diagram. One of them is the optimum (e.g., black). The classes along the optimal path determine the classes to which ω_i are assigned.

- To each transition corresponds a cost. For our case

- $$\hat{d}(\omega_{i_k}, \omega_{i_{k-1}}) = P(\omega_{i_k} | \omega_{i_{k-1}}).$$
  $$p(\underline{x}_k | \omega_{i_k})$$

- $$\hat{d}(\omega_{i_1}, \omega_{i_0}) \equiv P(\omega_{i_1}) p(\underline{x}_i | \omega_{i_1})$$

- $$\hat{D} = \prod_{k=1}^{N} \hat{d}(\omega_{i_k}, \omega_{i_{k-1}}) = p(X | \Omega_i) P(\Omega_i)$$

- Equivalently

$$\ln \hat{D} = \sum_{k=1}^{N} \ln \hat{d}(.,.) \equiv D = \sum_{k=1}^{N} d(.,.)$$

where,

$$d(\omega_{i_k}, \omega_{i_{k-1}}) = \ln \hat{d}(\omega_{i_k}, \omega_{i_{k-1}})$$

- Define the cost up to a node $,k,$

$$D(\omega_{i_k}) = \sum_{r=1}^{k} d(\omega_{i_r}, \omega_{i_{r-1}})$$

- Bellman's principle now states

$$\boxed{\begin{array}{l} D_{\max}(\omega_{i_k}) = \max_{i_{k-1}} \left[D_{\max}(\omega_{i_{k-1}}) + d(\omega_{i_k}, \omega_{i_{k-1}}) \right] \\ i_k, i_{k-1} = 1, 2, \ldots, M \end{array}}$$

$$D_{\max}(\omega_{i_0}) = 0$$

- The optimal path terminates at ω_{iN}^{*} :

$$\omega_{i_N}^{*} = \arg\max_{\omega_{i_N}} D_{\max}(\omega_{i_N})$$

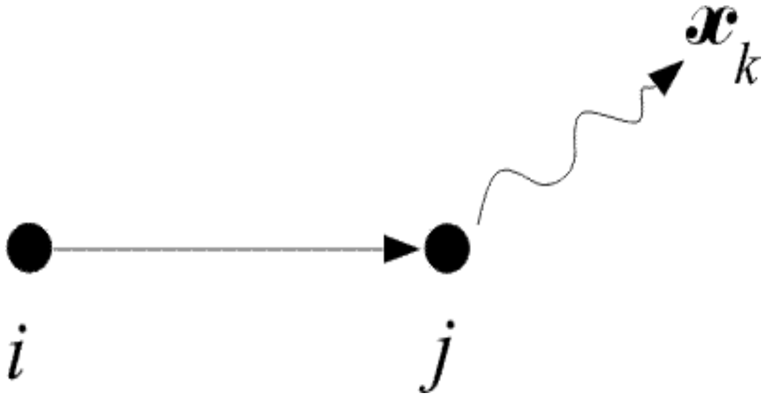
- Complexity $O\,(NM^2)$

❖ Hidden Markov Models

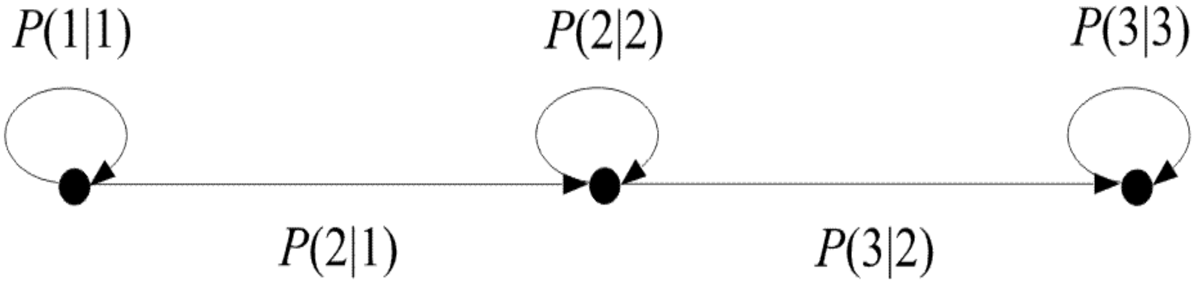
- Now we shall assume that states are not observable and can only be inferred from the training data

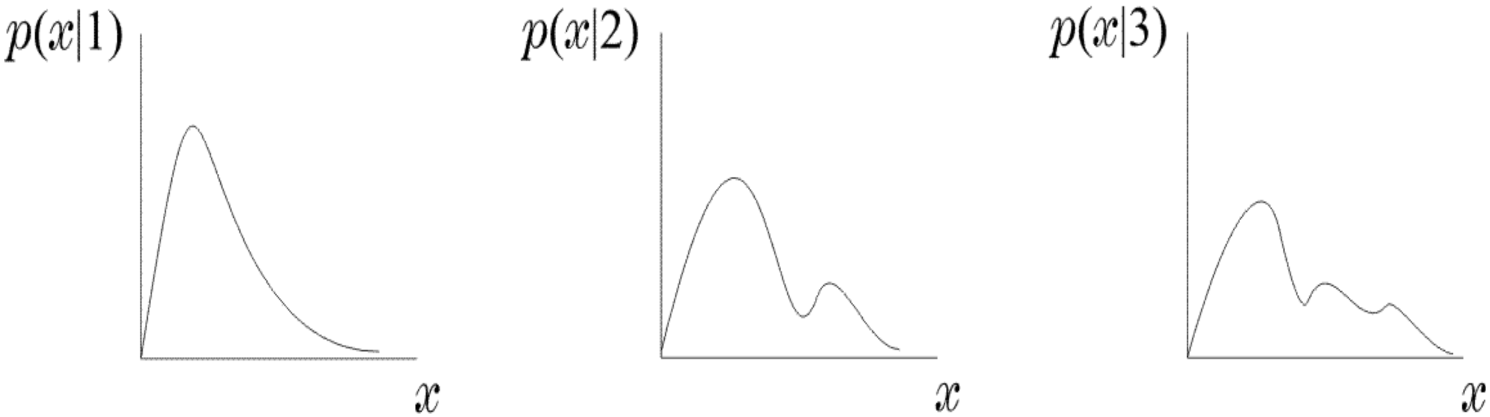
- Applications:
  - Speech and Music Recognition
  - OCR
  - Blind Equalization
  - Bioinformatics

- An HMM is a stochastic finite state automaton, that generates the observation sequence, $\underline{x}_1, \underline{x}_2, \ldots, \underline{x}_N$
- We assume that: The observation sequence is produced as a result of **successive** transitions between states, upon arrival at a state:



- This type of modeling is used for nonstationary stochastic processes that undergo distinct transitions among a set of different stationary processes.





- Examples of HMM:
  - The single coin case: Assume a coin that is tossed behind a curtain. All it is available to us is the outcome, i.e., H or T. Assume the two states to be:

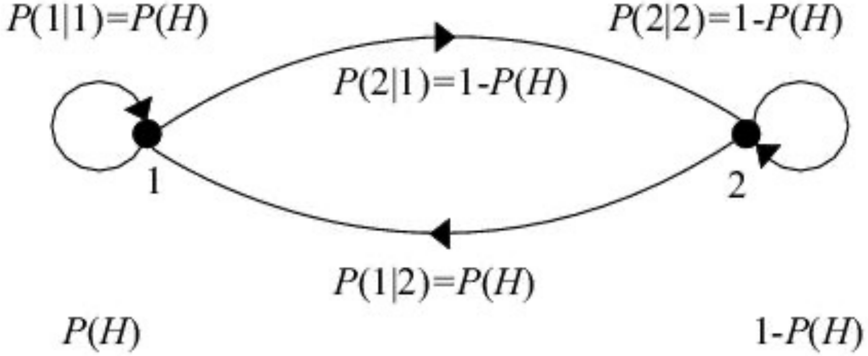
$$S = 1 \rightarrow H$$

$$S = 2 \rightarrow T$$

    This is also an example of a random experiment with observable states. The model is characterized by a single parameter, e.g., $P(H)$. Note that

$$P(1|1) = P(H)$$

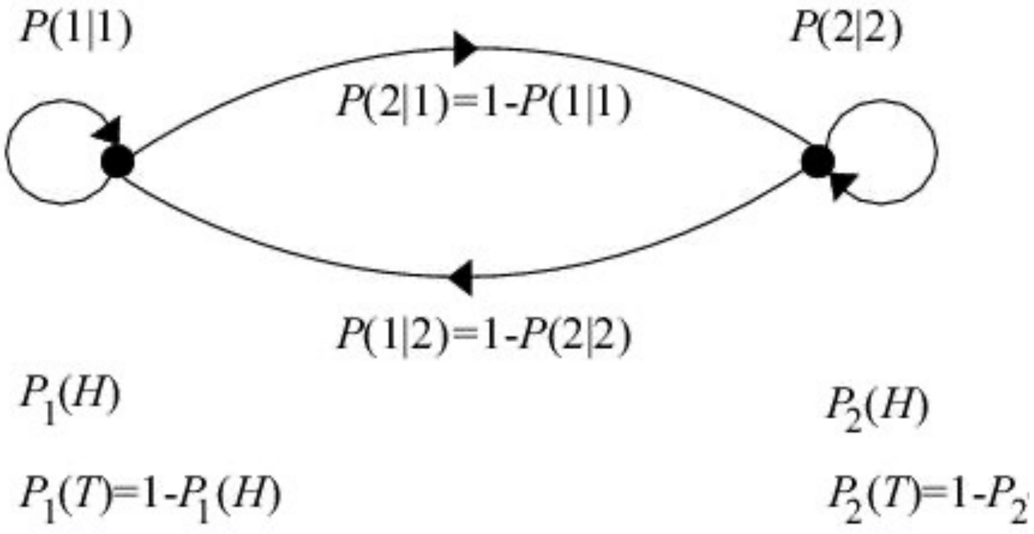
$$P(2|1) = P(\mathrm{T}) = 1 - P(H)$$



(a)

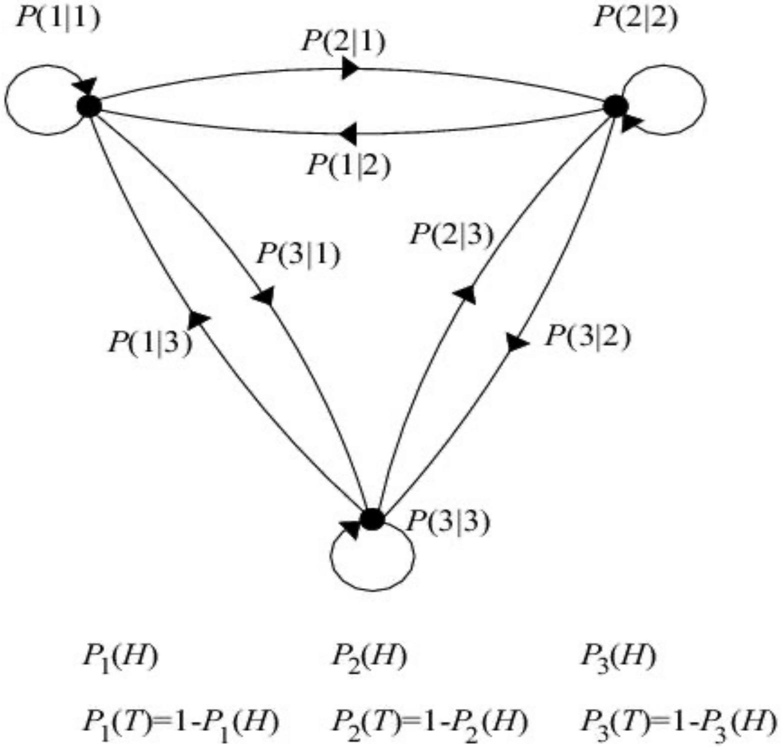
- The two-coins case: For this case, we observe a sequence of H or T. However, we have no access to know which coin was tossed. Identify one state for each coin. This is an example where states are not observable. H or T can be emitted from either state. The model depends on four parameters.

$$P_1(H), P_2(H),$$
$$P(1|1), P(2|2)$$



(b)

- The three-coins case example is shown below:



- Note that in all previous examples, specifying the model is equivalent to knowing:
  - The probability of each observation (H,T) to be emitted from each state.
  - The transition probabilities among states: $P(i|j)$.

- A general HMM model is characterized by the following set of parameters

  - K, number of states
  - $P(i|j), i, j = 1,2,...,K$
  - $p(\underline{x}|i), i = 1,2,...,K$
  - $P(i), i = 1,2,...,K$, initial state probabilities, $P(.)$

That is:

$$\boxed{S = \{P(i|j), p(\underline{x}|i), P(i), K\}}$$

- What is the problem in Pattern Recognition
  - Given M reference patterns, each described by an HMM, find the parameters, S, for each of them (training)
  - Given an unknown pattern, find to which one of the M, known patterns, matches best (recognition)

Recognition: Any path method

- Assume the M models to be known (M classes).
- A sequence of observations, X, is given.
- Assume observations to be emissions upon the arrival on successive states
- Decide in favor of the model S^* (from the M available) according to the Bayes rule

$$\boxed{S^* = \arg\max_S P(S|X)}$$

for equiprobable patterns

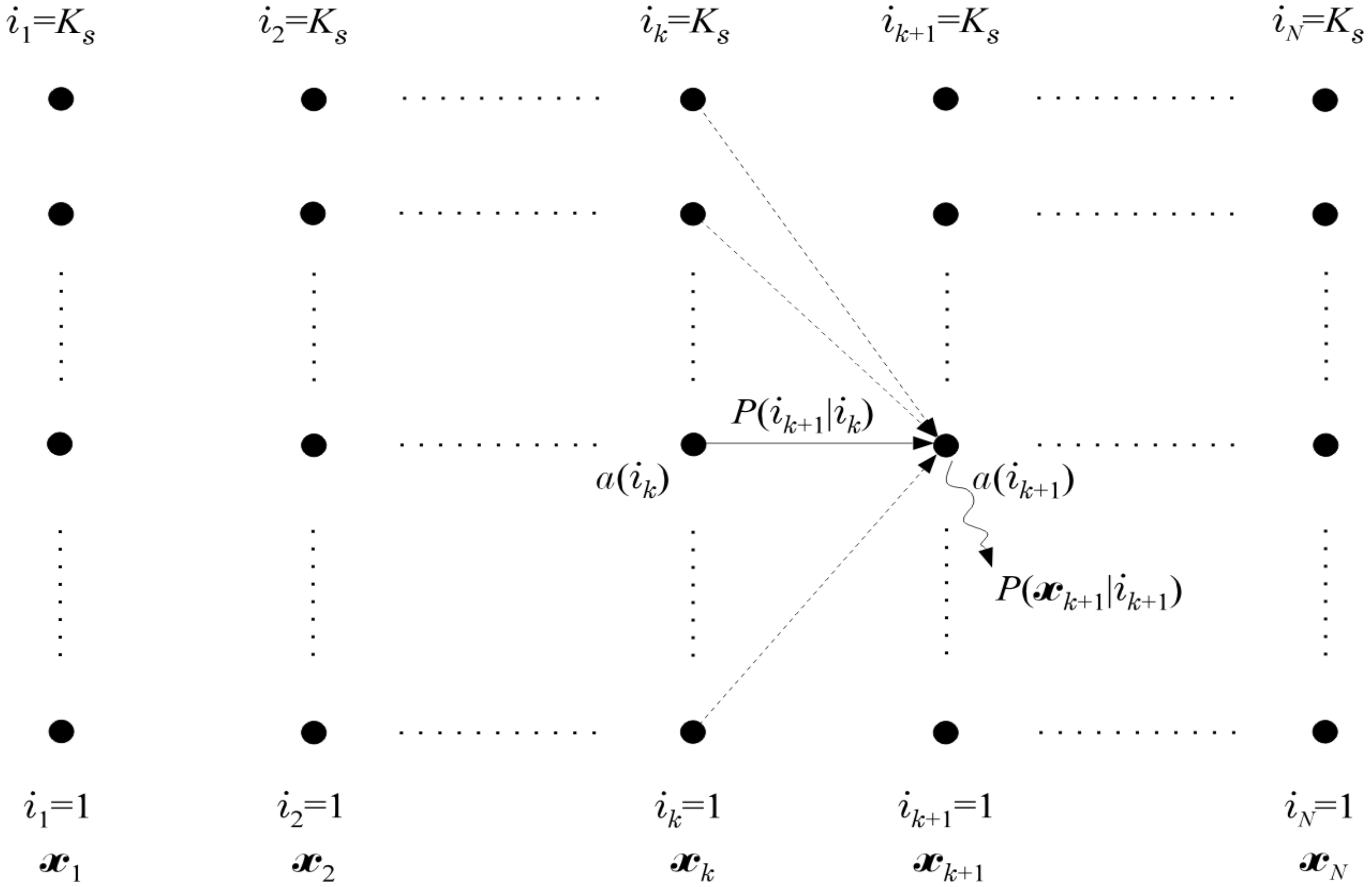
$$\boxed{S^* = \arg\max_S p(X|S)}$$

- For each model *S* there is more than one possible sets of successive state transitions Ω_i, each with probability $P(\Omega_i|S)$

Thus:
$$\boxed{\begin{aligned} P(X|S) &= \sum_i p(X,\Omega_i|S) \\ &= \sum_i p(X|\Omega_i,S)P(\Omega_i|S) \end{aligned}}$$

- For the efficient computation of the above DEFINE

  – $$\alpha(i_{k+1}) = p(\underline{x}_1,\ldots,\underline{x}_{k+1},i_{k+1}|S)$$
  $$= \underbrace{\sum_{i_k}\alpha(i_k)}_{\text{History}}\ \underbrace{P(i_{k+1}|i_k)\,p(\underline{x}_{k+1}|i_{k+1})}_{\text{Local activity}}$$



- Observe that

$$\boxed{P(X|S)=\sum_{i_N=1}^{K_S}\alpha(i_N)}$$

Compute this for each S

- Some more quantities

  - $$\begin{aligned}\beta(i_k) &= p(\underline{x}_{k+1}, \underline{x}_{k+2}, \ldots, \underline{x}_N \,|\, i_k, S) \\ &= \sum_{i_{k+1}} \beta(i_{k+1}) P(i_{k+1} \,|\, i_k)\, p(\underline{x}_{k+1} \,|\, i_{k+1})\end{aligned}$$

  - $$\begin{aligned}\gamma(i_k) &= p(\underline{x}_1, \ldots, \underline{x}_N, i_k \,|\, S) \\ &= \alpha(i_k)\beta(i_k)\end{aligned}$$

- Training
  - The philosophy:

    Given a training set X, known to belong to the specific model, estimate the unknown parameters of S, so that the **output** of the model, e.g.

    $$p(X|S) = \sum_{i_{N=1}}^{K_s} \alpha(i_N)$$

    to be maximized

- This is a ML estimation problem with missing data

- Assumption: Data $\underline{x}$ discrete

$$\underline{x} \in \{1,2,\ldots,r\} \Rightarrow p(\underline{x}|i) \equiv P(\underline{x}|i)$$

- Definitions:

- $$\xi_k(i,j) = \frac{\alpha(i_k = i)P(j|i)P(\underline{x}_{k+1}|j)\beta(i_{k+1} = j)}{P(X|S)}$$

- $$\gamma_k(i) = \frac{\alpha(i_k = i)\beta(i_k = i)}{P(X|S)}$$

- **The Algorithm**:
  - Initial conditions for all the unknown parameters. Compute $P(X|S)$
  - Step 1: From the current estimates of the model parameters reestimate the new model S from
    - $$\bar{P}(j|i) = \frac{\sum_{k=1}^{N-1} \xi_k(i,j)}{\sum_{k=1}^{N-1} \gamma_k(i)} \quad \left(= \frac{\#\,\text{of transitions from } i \text{ to } j}{\#\,\text{of transitions from } i} \right)$$
    - $$\boxed{\bar{P}_{\underline{x}}(r|i) = \frac{\sum_{k=1 \text{ and } \underline{x}\to r)}^{N} \gamma_k(i)}{\sum_{k=1}^{N} \gamma_k(i)}} \quad \left(= \frac{\text{at state } i \text{ and } \underline{x} = r}{\neq \text{of being at state } i} \right)$$
    - $$\boxed{\bar{P}(i) = \gamma_1(i)}$$

- Step 3: Compute $P(X|\overline{S})$. If $P(X|\overline{S}) - P(X|S) > \varepsilon,\ S = \overline{S}$ go to step 2. Otherwise stop

- Remarks:
  - Each iteration improves the model

    $$\overline{S}:\ P(X|\overline{S}) > P(X|S)$$

  - The algorithm converges to a maximum (local or global)
  - The algorithm is an implementation of the EM algorithm