Introduction to the basic concepts of the Portfolio Theory

This overview provides a brief description of the process of constructing an efficient risky Portfolio. We start the analysis by explaining why and how diversification reduces the variability of portfolio returns. Then, it is presented the process of constructing optimal risky portfolios.

1.1 Diversification and Portfolio Risk[1]

Suppose that our risky portfolio consists of only one stock. We need to specify the potential sources of riskiness of this simple portfolio. There are two main sources of uncertainty. The first source of risk has to do with the general economic conditions, such as interest rates, exchange rates, inflation rate and the business cycle. None of the above stated macroeconomic factors can be predicted with accuracy and all affect the rate of return of the portfolio. The second source of uncertainty is firm specific. Analytically, it has to do with the prospects of the firm, the management, the results of the research and development department of the firm, etc. In general, firm specific risk can be defined as the uncertainty that affects a specific firm without noticeable effects on other firms.

If now we decide to add another stock to our portfolio (let say for example stock 2), what will be the effect to the portfolio risk? The answer to this question depends on the relation between stock 1 and stock 2. If the firm specific risk of the two stocks differs (statistically speaking stock 1 and stock 2 are independent) then the portfolio risk will be reduced. Practically, the two opposite effects offset each other, which have as a result the stabilization of the portfolio return.

The relation between stock 1 and stock 2 in statistics is called correlation. Correlation describes how the returns of two assets move relative to each other through time[2]. The most well known way of measuring the correlation is the correlation coefficient (r). The correlation coefficient can range from –1 to 1.

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[1] Bodie, Kane and Marcus, *Essentials of Investments*, McGraw-Hill,2003, pp 169-170
[2] Mayes T., Shank T., *Financial Analysis with Microsoft Excel*, p. 364

Figure 1, illustrates two extremes situations: Perfect Positive correlation (r=1) and Perfect Negative correlation (r=-1).

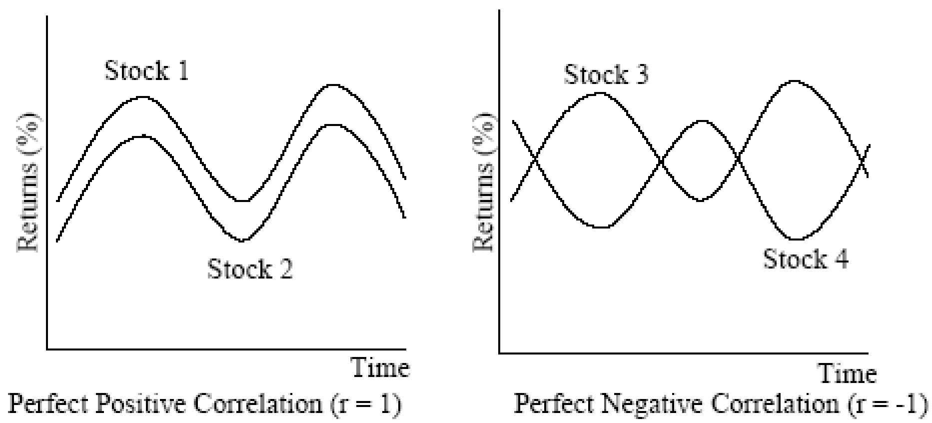

Figure 1

Another well-known way to measure the relation between any two stocks is the covariance. The covariance (cov(X,Y) = $\sigma_{X,Y}$) is calculated according to the following formulae:

$$\sigma_{X,Y} = \frac{1}{N}\sum_{t=1}^{N}(X_t - \overline{X})(Y_t - \overline{Y})$$

There is a relation between the correlation coefficient that we presented above, and the covariance. This relation is illustrated through the following formulae:

$$r_{X,Y} = \frac{\sigma_{X,Y}}{\sigma_X \sigma_Y}$$

The correlation coefficient is the same as the covariance, the only difference is that the correlation coefficient has been formulated in such way that it takes values from –1 to 1. Values of the correlation coefficient close to 1 mean that the returns of the two stocks move in the same direction, and values of the correlation coefficient close to –1 mean that the returns of the two stocks move in opposite directions.

A correlation coefficient $r_{x,y}$ = 0 means that the returns of the two stocks are independent. We make the assumption that our portfolio consists 50% of stock 1 and 50% of stock 2. On the left part of the Figure 1 because the returns of the two stocks are perfectly positively correlated the portfolio return is as volatile as if we owned either stock 1 or stock 2 alone. On the right part of the figure the stock 3 and stock 4 are perfectly negatively correlated. This way the volatility of return of stock 3 is cancelled out by the volatility of the return of stock 4. In this case, through diversification we achieve risk reduction.

Markowitz Mean-Variance (MV) formulation

Suppose 2 risky assets, whose rates of returns are given by the random variables r_1 and r_2

Portfolio Expected Return for N = 2:

$$\mu_p = E[r_p] = \sum_{t=1}^{N} x_t \ \mu_t = x_1 \ \mu_1 + x_2 \ \mu_2$$

Where μ_t = the average return of r_t

Portfolio Variance for N = 2:

$$\sigma_p^2 = var[r_p] = \sum_{t=1}^{N}\sum_{j=1}^{N} x_t \ x_j \ cov(r_t, r_j) = \sum_{t=1}^{N}\sum_{j=1}^{N} x_t \ x_j \ \sigma_{t,j} = x_1^2\sigma_1^2 + x_2^2\sigma_2^2 + 2x_1x_2\sigma_{1,2}$$

Negative covariance implies that the returns on assets x_t and x_j tend to move in opposite direction. If we invest in both securities at once the result is a portfolio that is less risky than holding either asset separately.

Assuming that both portfolio weights (stock 1 and 2) are positive then a positive covariance will tend to increase the portfolio variance and a negative covariance will tend to reduce the portfolio variance.

Correlation

Covariance is interesting because it is a quantitative measurement of the relationship between two variables. Correlation between two random variables,ρ(X,Y) is the covariance of the two variables normalized by the variance of each variable. This normalization cancels the units out and normalizes the measure so that it is always in the range [-1, 1]:

$$r_{\mathrm{X,Y}} = \frac{\sigma_{X,Y}}{\sigma_X \sigma_Y}$$

Thus, Portfolio Variance for N = 2:

The importance of the correlation coefficient is indicated by the following formulae:

$$\sigma_p^2 = x_1^2\sigma_1^2 + x_2^2\sigma_2^2 + 2x_1 x_2 r_{1,2}\sigma_1\sigma_2$$

The above formulae give us the portfolio variance for a portfolio of two securities 1 and 2. Where *x* are the weights for each security and $r_{1,2}$ is the correlation coefficient for the two securities.

The standard deviation of a two – securities portfolio is given by the formulae:

$$\sigma_p = \sqrt{x_1^2\sigma_1^2 + x_2^2\sigma_2^2 + 2x_1 x_2 r_{1,2}\sigma_1\sigma_2} = \sqrt{\sigma_p^2}$$

From the formulae, it is obvious that the lower the correlation coefficient $r_{1,2}$ between the securities, the lower the risk of the portfolio will be.

Obviously, if we continue to add securities that are negatively correlated into the portfolio the firm-specific risk will continue to reduce. Eventually, however even with a large number of negatively correlated stocks in the portfolio it is not possible to eliminate risk. This happens because all securities are subject to macroeconomic factors such as inflation rate, interest rates, business cycle, exchange rates, etc. Consequently, no matter how well we manage to diversify the portfolio it is still exposed to the general economic risk.

The following figure illustrates what we described in the previous page:

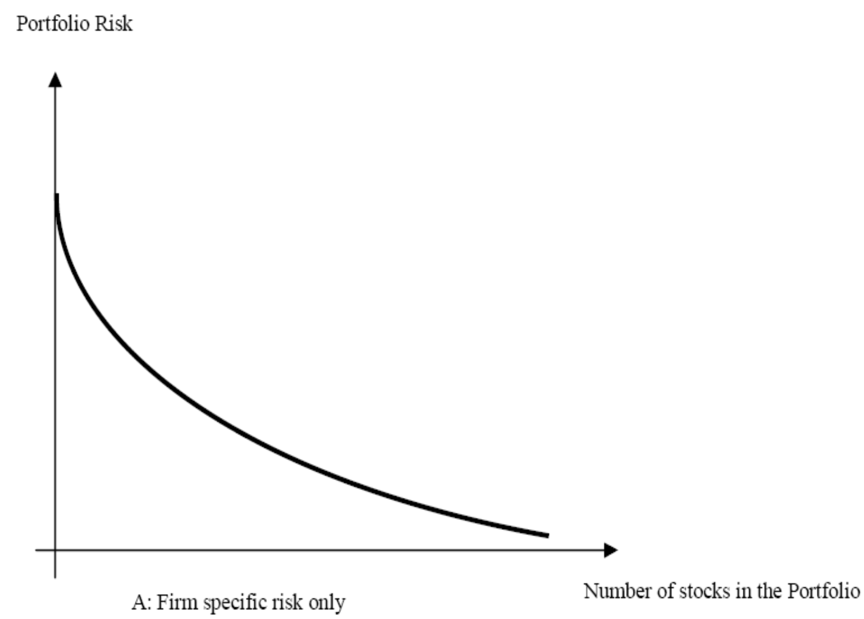

Figure 2

In figure 2 we can see that the firm specific risk can be eliminated if we add a large number of negatively correlated securities into the portfolio. The risk that can be eliminated by diversification except from firm specific risk is called unique risk, non systematic risk or diversifiable risk.

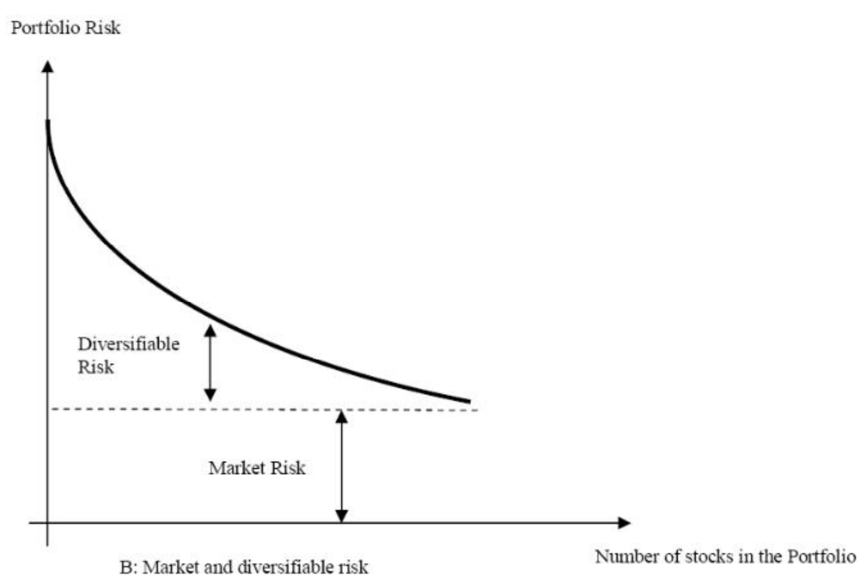

Figure 3

In figure 3 we can see that no matter how well diversified is the portfolio there is no way to get rid of the exposure of the portfolio to the macroeconomic factors. These factors related to the general economic risk are called market risk or systematic risk or non diversifiable risk.

1.2 The Portfolio Theory[3]

The expected returns μ_1 and μ_2 are the averages of the probability distributions of the possible returns.

The variances σ^2_1 and σ^2_2 are a measure of the uncertainty associated with these returns. The lower the variance the less risky considered a security.

The covariance $\sigma_{1,2}$ provides information about the dependence between returns. For example if $\sigma_{1,2} > 0$ then the returns of stock 1 and 2 move in the same direction; If $\sigma_{1,2} < 0$ the returns of stock 1 and 2 move in the opposite directions and finally if $\sigma_{1,2} = 0$ then the returns move independently.

The main question to be answered in any portfolio is:

How to select the weight of each particular security (i.e. the proportion of the portfolio invested in each particular stock) in order to achieve the optimum return for a given level of risk?

Let assume that X_1 denote the share of wealth invested in stock 1 and X_2 denote the share of wealth invested in stock 2. Since all wealth is invested in the two stock 1 and stock 2 it follows that $X_1 + X_2 = 1$.

The variance of a portfolio consisting of two stocks (1 and 2) is the weighted average of the variances of the individual stocks plus two times the product of the portfolio weights times the covariance between the securities.

Assuming that both portfolio weights (stock 1 and 2) are positive then a positive covariance will tend to increase the portfolio variance and a negative covariance will tend to reduce the portfolio variance.

[3] Zivot Eric, *Introduction to Financial Econometrics, Chapter 4 Introduction to Portfolio Theory*, Univ. of Washington, 2000

Assumptions of Markowitz Portfolio Theory

(1) Investors are rational and behave in a manner as to maximise their utility with a given level of income or money.

(2) Investors have free access to fair and **correct information*** on the returns and risk.

(3) The markets are efficient and absorb the information quickly and perfectly.

(4) Investors are risk averse and try to minimise the risk and maximise return.

(5) Investors base decisions on expected returns and variance or standard deviation of these returns from the mean.

(6) Investors choose higher returns to lower returns for a given level of risk.

A portfolio of assets under the above assumptions is considered efficient if no other asset or portfolio of assets offers a higher expected return with the same or lower risk or lower risk with the same or higher expected return.

(*) Efficient Market Hypothesis (EMH)

The Efficient Market Hypothesis (EMH) essentially says that all known information about investment securities, such as stocks, is already factored into the prices of those securities1. Therefore, assuming this is true, no amount of analysis can give an investor an edge over other investors, collectively known as "the market."

EMH does not require that investors be rational; it says that individual investors will act randomly, but as a whole, the market is always "right." In simple terms, "efficient" implies "normal." For example, an unusual reaction to unusual information is normal. If a crowd suddenly starts running in one direction, it's normal for you to run in that direction as well, even if there isn't a rational reason for doing so.

Defining the Forms of EMH:

There are three forms of EMH: weak, semi-strong, and strong. Here's what each says about the market.

(1) Weak Form EMH: Suggests that all past information is priced into securities. Fundamental analysis of securities can provide an investor with information to produce returns above market averages in the short term, but there are no "patterns" that exist. Therefore, fundamental analysis does not provide long-term advantage and technical analysis will not work.

(2) Semi-Strong Form EMH: Implies that neither fundamental analysis nor technical analysis can provide an advantage for an investor and that new information is instantly priced in to securities.

(3) Strong Form EMH. Says that all information, both public and private, is priced into stocks and that no investor can gain advantage over the market as a whole. Strong Form EMH does not say some investors or money managers are incapable of capturing abnormally high returns because that there are always outliers included in the averages.

EMH does not say that no investors can outperform the market; it says that there are outliers that can beat the market averages; however, there are also outliers that dramatically lose to the market. The majority is closer to the median. Those who "win" are lucky and those who "lose" are unlucky.

Limitations of Markowitz model

(1) The model assumes that asset correlations are static or linear. In reality, asset correlations move dynamically, changing with the market cycles. During the global financial crisis, asset correlations approached almost to 1, so if anything, diversification seemed to have insignificant impacts on the portfolios.

(2) MVO assumes normality in return distributions. Therefore, it does not factor in extreme market moves which tend to make returns distributions either skewed, fat tailed or both. Without optimizing the portfolio for asset that may actually have skewed distributions or fat tailed, MVO could lead to a riskier allocation that is intended.

(3) It is difficult to forecast asset returns with accuracy using historical data, which tends to be a poor forecasting source. As return estimations have a much larger impact on MVO asset allocations, small changes in return assumptions can lead to inefficient portfolios.

(4) Even with the most careful planning and portfolio construction, past performance is never a guarantee for future results.

Real World constraints that must be applied to the original model

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How to implement additional constraints to the proposed model – extensions

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