Portfolio Optimization under Different Risk Measures: A Three-Dimensional Encoding Multi-objective Evolutionary Algorithm Approach

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Motivation

- The mean-variance (M-V) framework is traditionally used for expressing the compromise between portfolio's expected return and its associated risk.
- However, the current research indicates that the portfolio's risk is better quantified with the assistance of alternative risk measures
- Moreover, another issue of concern is the amount of processing time required for the portfolio optimization process.
- A common issue with the existing techniques is the excessive processing time required, especially for large instances of the portfolio optimization problem.

Research Objectives

- To develop a multi-objective evolutionary algorithm (MOEA) that its processing time is less susceptible to the size of the examined test instance.
- To examine different risk measures that present better properties than the classical mean-variance (M-V) framework.
- Ideally, the proposed algorithm should find useful application to the solution of other complex problems as well, beyond the portfolio optimization problem.

Problem Definition

- The portfolio optimization problem is formulated as a bi-objective problem $f(w)=(f_1(w),f_2(w))$ where two conflicting objectives f_1 , f_2 should be satisfied at any time.

- The first objective (f_1) which is maximized corresponds to the return of the portfolio and the second objective (f_2) which is minimized corresponds to the risk of the portfolio.

The Mean – Variance portfolio optimization problem

- The portfolio optimization problem is formulated as a bi-objective problem $f(w)=(f_1(w),f_2(w))$ where two conflicting objectives f_1 , f_2 should be satisfied at any time.

- Maximize portfolio return: $f_1(w)=\sum_{i=1}^{N} w_i \overline{r_i}$

- Minimize portfolio variance: $f_2(w)=\sum_{i=1}^{N}\sum_{j=1}^{N} w_i w_j \sigma_i \sigma_j \rho_{ij}$

Assumptions of the mean-variance portfolio optimization framework that have been criticised

- The financial markets are informationally efficient
- Asset returns are jointly normally distributed variables
- Correlations between assets are fixed and constant
- The Mean-Variance model does not consider taxes or transaction cost
- Any investor can lend and borrow an unlimited amount at the risk free rate of interest
- All assets are divisible into lots of any size

The Mean – Semiariance portfolio optimization problem

- According to the Estrada [1] the portfolio semivariance is approximated by the following relationship:

$$SV_{pB} \approx \sum_{i=1}^{N} \sum_{j=1}^{N} w_i w_j SC_{ijB}$$

- Where SC_{ijB} is the sample semicovariance:

$$SC_{ijB} \approx \frac{1}{T} \sum_{t=1}^{T} \left[\min(R_{it} - B, 0) * \min(R_{jt} - B, 0) \right]$$

- [1] Estrada, J. (2007) Mean-Semivariance Optimization: A Heuristic Approach. The Journal of Applied Finance, Spring / Summer, 57-71.

The Mean – Semivariance portfolio optimization problem

- Thus, the Mean – Semivariance (M-SV) portfolio optimization problem is formulated as follows:

- Maximize portfolio return: $f_1(w) = \sum_{i=1}^{N} w_i \overline{r_i}$

- Minimize portfolio semivariance: $f_2(w) = \sum_{i=1}^{N} \sum_{j=1}^{N} w_i w_j SC_{ijB}$

The Mean – Value at Risk portfolio optimization problem

- Thus, the Mean – Value at Risk (VaR) portfolio optimization problem is formulated as follows:

- Maximize portfolio return: $\mathrm{f}_1(\mathrm{w}) = \sum_{i=1}^{N} w_i \overline{r_i}$

- Minimize portfolio VaR: $\mathrm{f}_2(\mathrm{w}) = -\inf\left\{ z_t(w) \mid \sum_{t=1}^{T} p_i \geq \alpha \right\}$

- Where returns $z_t(w)$ are placed in an ascending order, i.e.

$z_1(w) \leq z_2(w) \leq \ldots \leq z_T(w)$

The Mean – Conditional Value at Risk portfolio optimization problem

- Thus, the Mean – Conditional Value at Risk (CVaR) portfolio optimization problem is formulated as follows:

- Maximize portfolio return: $f_1(w) = \sum_{i=1}^{N} w_i \overline{r_i}$

- Min portfolio CVaR: $f_2(w) = -E\{z_t(w) \mid z_t(w) < -VaR_a(w)\}$

The Mean – Mean Absolute Deviation portfolio optimization problem

- Thus, the Mean – Mean Absolute Deviation (MAD) portfolio optimization problem is formulated as follows:

- Maximize portfolio return: $f_1(w) = \sum_{i=1}^{N} w_i \overline{r_i}$

- Min portfolio CVaR: $f_2(w) = \frac{1}{T} \sum_{t=1}^{T} \left| \sum_{j=1}^{N} (r_{jt} - \overline{r_j}) w_j \right|$

Practical constraints imposed to the portfolio optimization problem

- Budget constraint $\sum_{i=1}^{N} w_i = 1$
- Non-negativity constraint: $0 \le w_i \le 1, \qquad i = 1,2,\ldots,N$
- Cardinality constraint: $\mathrm{K}_{\min} \le \sum_{i=1}^{N} q_i \le \mathrm{K}_{\max},$

  $where \quad q_i = 1 \quad for \;\; w_i > 0 \quad and \quad q_i = 0 \quad for \;\; w_i = 0$

- Floor and ceiling constraint: $l_i \le w_i \le u_i, \qquad \forall i = 1,2,\ldots,N$

Previous work

- Hybrid representation scheme [2]:
- Binary vector: $\Delta = \{z_1,\ldots,z_N\}, \quad z_i = \{0,1\}, \quad i = 1,2,\ldots,N$
- Real-valued vector: $W = \{w_1,\ldots,w_N\}, \quad 0 \le w_i \le 1, \quad i = 1,\ldots,N$

- [2] Streichert, F., Ulmer, H. and Zell, A. (2004) Evaluating a hybrid encoding and three crossover operators on the constrained portfolio selection problem. Proceedings of the Congress on Evolutionary Computation (CEC 2004), Portland, Oregon, (2004), pp. 932-939.

The proposed encoding scheme

- The proposed three-dimensional encoding scheme:
- Integer-valued vector: $\mathrm{A} = \{\alpha_1, \ldots, \alpha_j\}, \quad j \in \{1, \ldots, \mathrm{N}\}, \quad |\mathrm{A}| = \mathrm{K}_{\max}$
- Real-valued vector : $W = \{w_1, \ldots, w_i\}, \quad 0 \leq w_i \leq 1, \quad i = 1, \ldots, \mathrm{K}_{\max}$
- Binary-valued vector: $\Delta = \{z_1, \ldots, z_i\}, \quad z_i = \{0,1\}, \quad i = 1,2, \ldots, \mathrm{K}_{\max}$
- where $\sum_{i=1}^{\mathrm{K}_{\max}} w_i = 1 \;\; and \;\; |W| = \mathrm{K}_{\max} \;\; , \;\; |\Delta| = \mathrm{K}_{\max}$

The proposed mutation operator

- We apply polynomial mutation (PLM) for mutating the real-valued vector (W)
- We apply bit-flip (BF) mutation operator for mutating the binary-valued solution vector (Δ)
- Because the integer-valued vector (A) of the proposed representation scheme is restricted by the maximum cardinality (K_{max}) we need to incorporate into the proposed genetic operators an efficient alteration mechanism that will contribute to the updating process of the assets' solution vector

The proposed mutation operator

```
Begin
 for i=0 to P;   // where P is the population size
      for z=0 to K_max ;  // where K_max is the maximum cardinality
        rand --> [0, 1];
          if (rand <= P_m) then
            rand_asset --> [1, N]; //where N is the available pool of assets, e.g. for port5, N = 225
              A_c = A_p .setValue(z, rand_asset);   //A_c is the child integer-valued vector
                                                   // A_p is the parent integer-valued vector
          endif
      endfor
 endfor
```

Explanation of symbols

P_m is a user-specified parameter named mutation probability

The proposed recombination operator

- We apply simulated binary crossover (SBX) for crossing the real-valued vector (W)
- We apply single point crossover (SPX) for crossing the binary-valued solution vector (Δ)
- For the integer-valued (A) vector, due to its special role to the updating process of the assets' solution vector we had to introduce a specially designed recombination operator for facilitating this process.

The proposed recombination operator

```
Begin
 for i=0 to P;  // where P is the population size
     for z=0 to K_max ; // where K_max is the maximum cardinality
        rand ---> [0, 1];
        if (rand <= P_c) then
            A_p^(1) = (a_1, a_2, ..., a_Kmax);
            A_p^(2) = (a_1, a_2, ..., a_Kmax);
            cross_point ---> [0, K_max - 1];
            for q = cross_point to K_max ;
               a_p^(2) = A_p^(2).getValue(q);
               A_c^(1) = A_p^(1).setValue(q, a_p^(2));
               a_p^(1) = A_p^(1).getValue(q);
               A_c^(2) = A_p^(2).setValue(q, a_p^(1));
            endfor
        endif
     endfor
 endfor
```

Explanation of symbols

$A_p^{(1)}$ is the parent integer-valued vector 1. $a_p^{(1)}$ is the integer-valued parent decision variable that corresponds to $A_p^{(1)}$

The test problems

Problem Formulation	Problem Name	Stock Market Index	Assets	Source
Mean-Variance	port5	Nikkei 225	225	OR-Library
Mean-Variance	port6	S&P 500	457	OR-Library
Mean-Variance	port7	Russell 2000	1317	OR-Library
Mean -Semivariance	FTSE-100	FTSE 100	92	our own dataset
Mean -VaR	SP500	S&P 500	442	Data in Brief
Mean -CVaR	SP500	S&P 500	442	Data in Brief
Mean-MAD	SP500	S&P 500	442	Data in Brief

Configuration and control parameters of the examined algorithms

- In all tests with the examined algorithms we use binary tournament as selection operator.
- For the real valued (W) solution vector of the Three-Dimensional Encoding Multi-objective Evolutionary Algorithm (TDMEA) we use, simulated binary crossover (SBX) and polynomial mutation (PLM), as crossover and mutation operator, respectively.
- For the binary-valued (Δ) solution vector of TDMEA we use single-point crossover (SPX) and bit-flip (BF) mutation operator, as recombination and mutation operator respectively.
- Finally for mutating and crossing the integer-valued (A) solution vector of TDMEA we use the proposed mutation and recombination operators as appear in slide 17 and 19 respectively.

Configuration and control parameters of the examined algorithms

- For the Non-nominated Sorting Genetic Algorithm II (NSGAII) we used its typical configuration with simulated binary crossover (SBX) and polynomial mutation (PLM) for the real-valued solution vector and single-point crossover (SPX) and bit-flip (BF) mutation operator, for the binary-valued solution vector respectively.
- The settings for the Strength Pareto Evolutionary Algorithm 2 (SPEA2) are the following: simulated binary crossover (SBX) and polynomial mutation (PLM) as recombination and mutation operator respectively of the real-valued solution vector and single-point crossover (SPX) and bit-flip (BF) mutation operator as recombination and mutation operator respectively of the binary-valued solution vector.
- All algorithms have been programmed in Java and run on a 3.30 GHz, Intel Core (TM) i5-4590 CPU machine, Windows 10 Pro with 4 GB RAM.

Configuration and control parameters of the examined algorithms

- We set crossover probability to $P_c = 0.9$ and mutation probability to $P_m = 0.1$ as control parameters for both the real and the binary solutions vectors of all examined algorithms.

- We set $P_c = 0.9$ and $P_m = 0.1$ as control parameters for the integer solution vector of TDMEA.

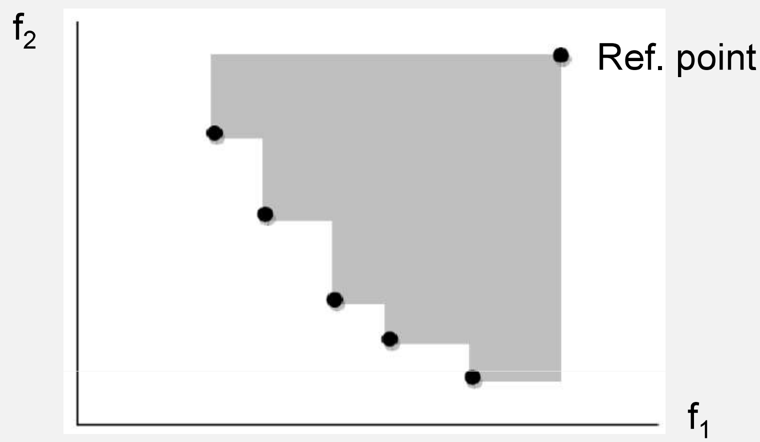
- We set the distribution index for the crossover operator to $\eta_c = 20$ and respectively for the mutation operator to $\eta_m = 20$ for all examined algorithms. The aforementioned values for the control parameters resulted after the required fine tuning of the examined algorithms.

- Finally, we employ a population size of 100 individuals and a maximum of 100,000 function evaluations with 20 runs for all examined algorithms.

Indicators for evaluating algorithms' performance

- ## Hypervolume indicator (HV)

Hypervolume can be described as the n-dimensional space that is contained by a solution set, relative to some reference point. As reference point is taken the worst known value in each objective. Hypervolume (HV) summarize in a single unary value information regarding the spread of the solutions along the Pareto front and the distance of the set from the Pareto-optimal front. Hypervolume indicator is maximized if the set of solutions contains all Pareto optimal points of the examined multi-objective optimization problem. Hypervolume (HV) is said to be Pareto compliant meaning that the hierarchy it provides between the candidate approximate solutions complies with the Pareto optimality conditions.



Hypervolume of a bi-objective minimization problem

Indicators for evaluating algorithms' performance

- ## Inverted Generational Distance (IGD)

Inverted Generational Distance (IGD) measures the closeness of the obtained solution set to the true Pareto optimal set and is given by the following relationship:

$$IGD(P,S) = \frac{\left(\sum_{i=1}^{|P|} d_i^q\right)^{1/q}}{|P|}$$

where d_i is the minimum Euclidean distance between an approximate solution ($\vec{s} \in S$) to the closest solution in the true Pareto optimal Frontier (P). The IGD metric is able to provide a measure for both convergence and diversity and takes a zero value when all the generated solutions are in the Pareto front.

Indicators for evaluating algorithms' performance

- Epsilon additive Indicator ($I_{\varepsilon+}$)

The epsilon additive indicator of an approximation set A ($I_{\varepsilon+}$) provides the minimum factor ε by which each point in the real front R can be added such that the resulting transformed approximation set is dominated by A.

The additive epsilon indicator between two approximations A and B of the Pareto set can be described as the smallest epsilon value that allows all the solutions in B to be ε-dominated by at least one solution in A:

$$I_{\varepsilon+}(\mathrm{A},\mathrm{B}) = \min_{\varepsilon \in \Re^{+}} \forall y \in B \ \ \exists \ \ x \in A \,|\, x \leq_{\varepsilon+} y$$

The smaller the epsilon value the better the examined solution.

Experimental Results

- The Table below presents the results for the cardinality constrained portfolio optimization problem under different risk measures. Specifically, it presents the mean, standard deviation (STD), median and interquartile range (IQR) of all the independent runs carried out for Hypervolume (HV), Inverted Generalization Distance (IGD) and Epsilon indicator respectively.

	Problem: Port5 Mean–Variance			Problem: Port6 Mean–Variance		
	TDMEA	NSGAII	SPEA2	TDMEA	NSGAII	SPEA2
HV. Mean and Std	7.85e-01 9.7e-03	7.21e-01 1.7e-02	7.28e-01 2.6e-02	7.11e-01 5.9e-03	6.47e-01 3.0e-02	6.63e-01 1.6e-02
HV. Median and IQR	7.85e-01 1.2e-02	7.23e-01 2.5e-02	7.36e-01 2.5e-02	7.10e-01 5.0e-03	6.49e-01 3.4e-02	6.66e-01 2.1e-02
IGD. Mean and Std	3.11e-04 4.0e-05	1.27e-02 3.9e-03	1.25e-02 3.2e-03	5.84e-04 5.9e-05	2.18e-02 4.6e-03	1.81e-02 4.7e-03
IGD. Median and IQR	3.09e-04 4.0e-05	1.32e-02 6.1e-03	1.26e-02 4.0e-03	5.83e-04 2.1e-05	2.24e-02 4.6e-03	1.85e-02 5.4e-03
EPSILON. Mean and Std	1.37e-04 2.5e-05	5.17e-04 8.2e-05	4.89e-04 1.2e-04	1.21e-03 1.1e-04	4.17e-03 1.1e-03	3.65e-03 7.3e-04
EPSILON. Median and IQR	1.35e-04 3.5e-05	4.81e-04 1.4e-04	4.57e-04 1.5e-04	1.21e-03 1.3e-04	4.23e-03 1.5e-03	3.76e-03 1.1e-03

	Problem: Port7 Mean–Variance			Problem: FTSE-100 Mean - Semivariance		
	TDMEA	NSGAII	SPEA2	TDMEA	NSGAII	SPEA2
HV. Mean and Std	6.94e-01 6.1e-03	5.60e-01 4.2e-02	5.18e-01 4.7e-02	6.16e-01 2.8e-03	5.96e-01 7.2e-03	5.90e-01 8.0e-03
HV. Median and IQR	6.93e-01 9.0e-03	5.66e-01 8.3e-02	5.22e-01 8.2e-02	6.17e-01 4.5e-03	5.95e-01 1.1e-02	5.91e-01 8.2e-03
IGD. Mean and Std	7.71e-04 1.3e-04	2.30e-02 5.8e-03	2.82e-02 3.4e-03	4.73e-04 3.3e-05	2.09e-03 4.7e-04	1.99e-03 5.3e-04
IGD. Median and IQR	7.68e-04 3.0e-04	2.41e-02 1.3e-02	2.77e-02 4.7e-03	4.69e-04 4.6e-05	1.93e-03 5.6e-04	1.99e-03 4.7e-04
EPSILON. Mean and Std	5.10e-03 1.0e-03	1.85e-02 4.6e-03	2.27e-02 3.5e-03	5.11e-05 7.8e-06	3.36e-04 9.3e-05	3.11e-04 9.6e-05
EPSILON. Median and IQR	5.34e-03 1.7e-03	1.91e-02 1.0e-02	2.25e-02 5.4e-03	5.20e-05 8.0e-06	3.32e-04 1.7e-04	3.11e-04 1.4e-04

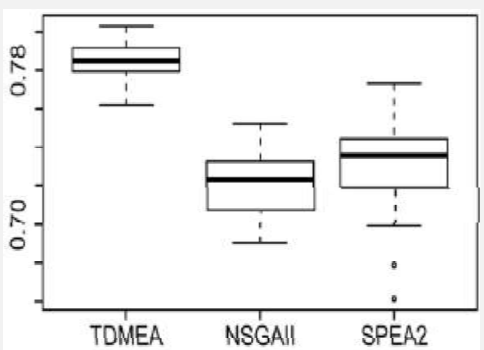
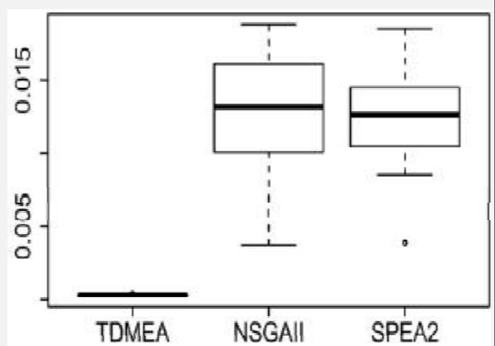
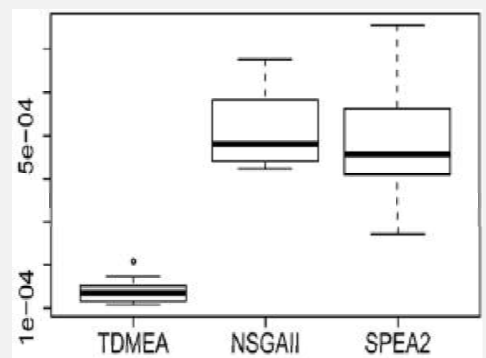
Experimental Results

	Problem: SP500 Mean – VaR TDMEA	NSGAII	SPEA2	Problem: SP500 Mean – CVaR TDMEA	NSGAII	SPEA2
HV. Mean and Std	6.28e-01 3.7e-02	3.79e-01 7.7e-02	3.23e-01 9.6e-02	5.80e-01 2.6e-02	4.38e-01 3.8e-02	4.37e-01 3.4e-02
HV. Median and IQR	6.24e-01 2.3e-02	3.79e-01 8.3e-02	2.99e-01 1.4e-01	5.81e-01 4.6e-02	4.21e-01 7.0e-02	4.42e-01 4.3e-02
IGD. Mean and Std	1.08e-03 1.6e-04	2.74e-02 6.9e-03	2.44e-02 8.7e-03	1.06e-03 2.2e-04	1.30e-02 2.4e-03	1.52e-02 4.8e-03
IGD. Median and IQR	1.06e-03 1.7e-04	2.67e-02 8.2e-03	2.35e-02 1.2e-02	1.10e-03 3.9e-04	1.28e-02 1.5e-03	1.48e-02 5.7e-03
EPSILON. Mean and Std	6.68e-03 1.0e-03	8.44e-03 9.0e-04	9.38e-03 1.8e-03	5.23e-03 1.3e-03	8.79e-03 8.7e-04	8.87e-03 1.4e-03
EPSILON. Median and IQR	6.66e-03 1.8e-03	8.66e-03 1.3e-03	9.66e-03 2.4e-03	5.05e-03 1.8e-03	8.89e-03 1.4e-03	8.45e-03 1.8e-03

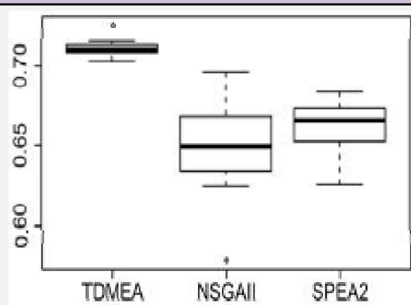
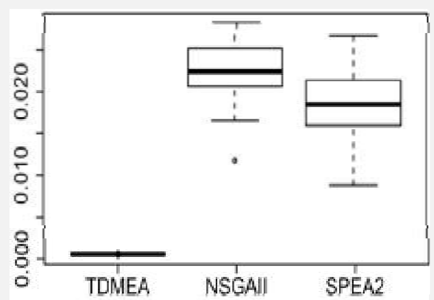
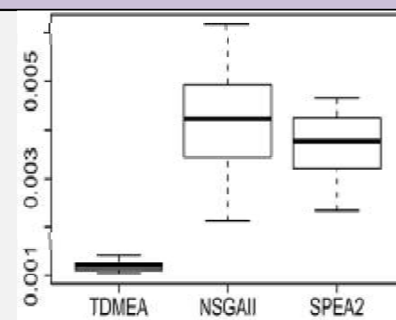
	Problem: SP500 Mean – MAD TDMEA	NSGAII	SPEA2
HV. Mean and Std	6.90e-01 3.7e-03	5.91e-01 1.4e-02	5.85e-01 1.3e-02
HV. Median and IQR	6.90e-01 6.0e-03	5.88e-01 1.8e-02	5.93e-01 2.1e-02
IGD. Mean and Std	5.95e-04 4.0e-05	7.87e-03 1.8e-03	8.54e-03 1.9e-03
IGD. Median and IQR	5.98e-04 6.0e-05	7.48e-03 3.8e-03	8.50e-03 3.4e-03
EPSILON. Mean and Std	3.01e-03 6.3e-04	9.69e-03 2.2e-03	1.02e-02 1.7e-03
EPSILON. Median and IQR	3.25e-03 1.1e-03	1.01e-02 4.2e-03	1.00e-02 1.3e-03

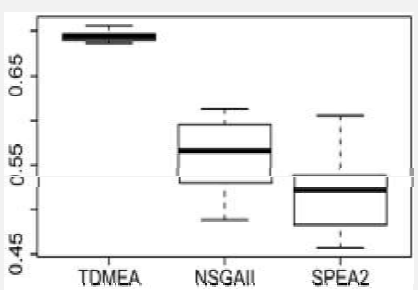
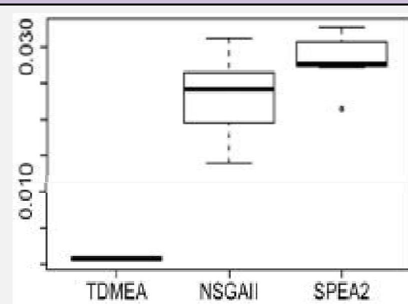
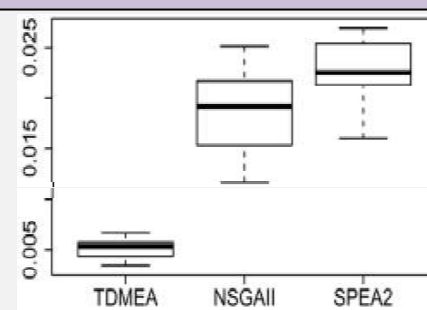
Experimental Results

- The Table below use boxplots to visualize the performance of the proposed three-dimensional encoding multiobjective evolutionary algorithm (TDMEA) against two well-known MOEAs, namely NSGAII and SPEA2, for HV, IGD and Epsilon performance indicators. Boxplots provide a simple graphical mean for comparing data sets that present information from a five number summary: minimum, maximum, median, first quartile and third quartile.

Problem: Port5 Mean–Variance		
HV	IGD	EPSILON
		

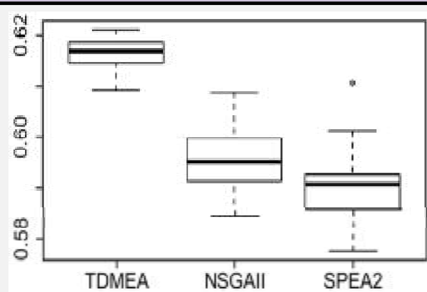
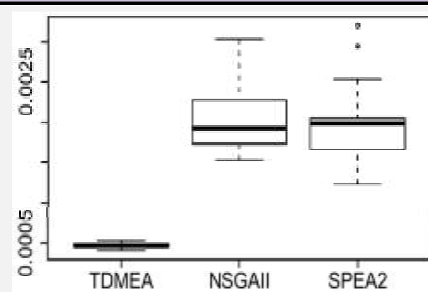
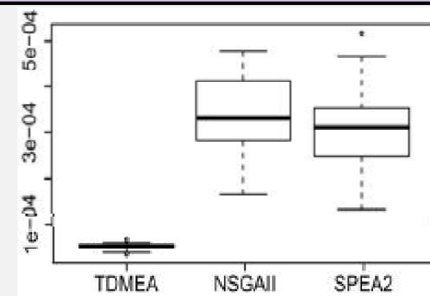
Experimental Results

Problem: Port6 Mean–Variance		
HV	IGD	EPSILON
		

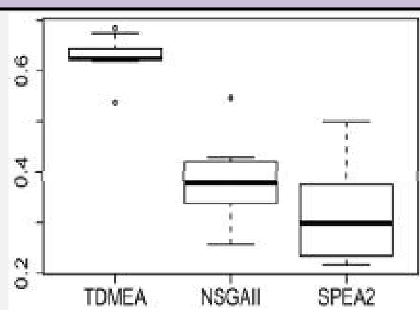
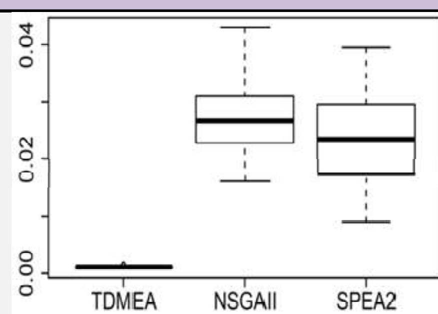
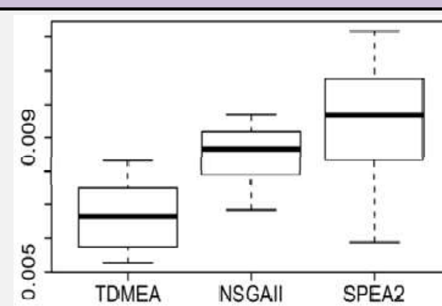
Problem: Port7 Mean–Variance		
HV	IGD	EPSILON
		

Experimental Results

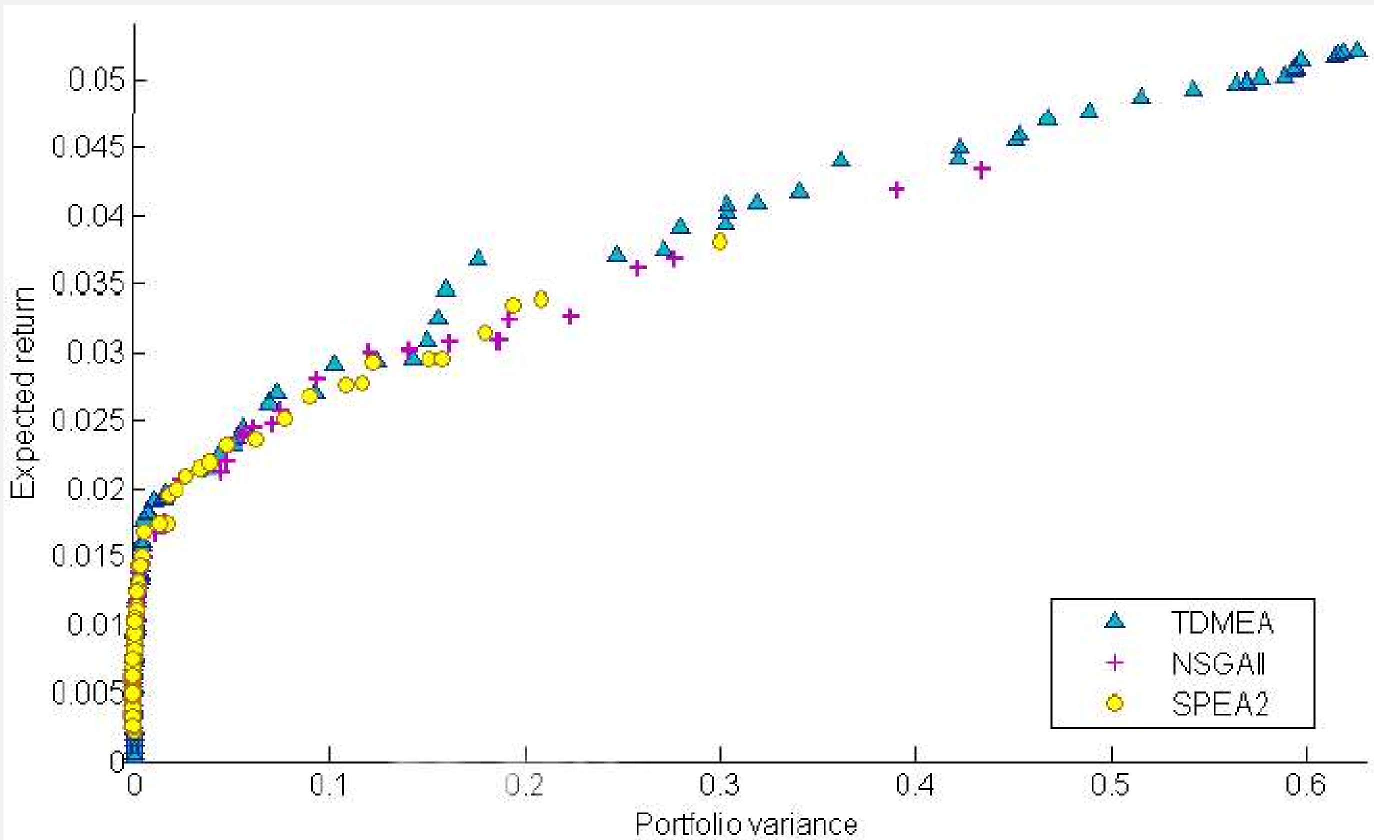
Problem: FTSE-100
Mean - Semivariance

HV	IGD	EPSILON
		

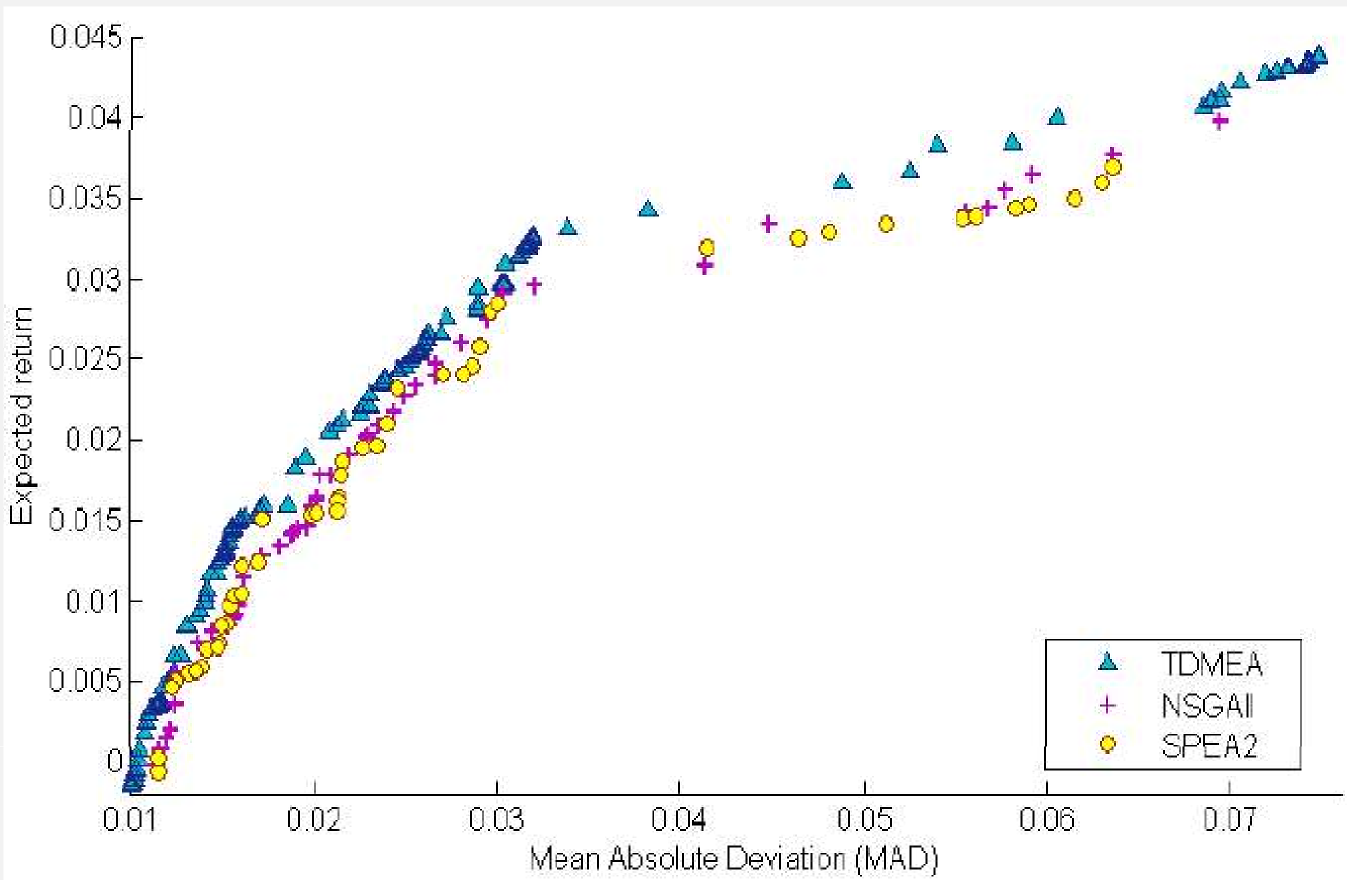
Problem: SP500
Mean – VaR

HV	IGD	EPSILON
		

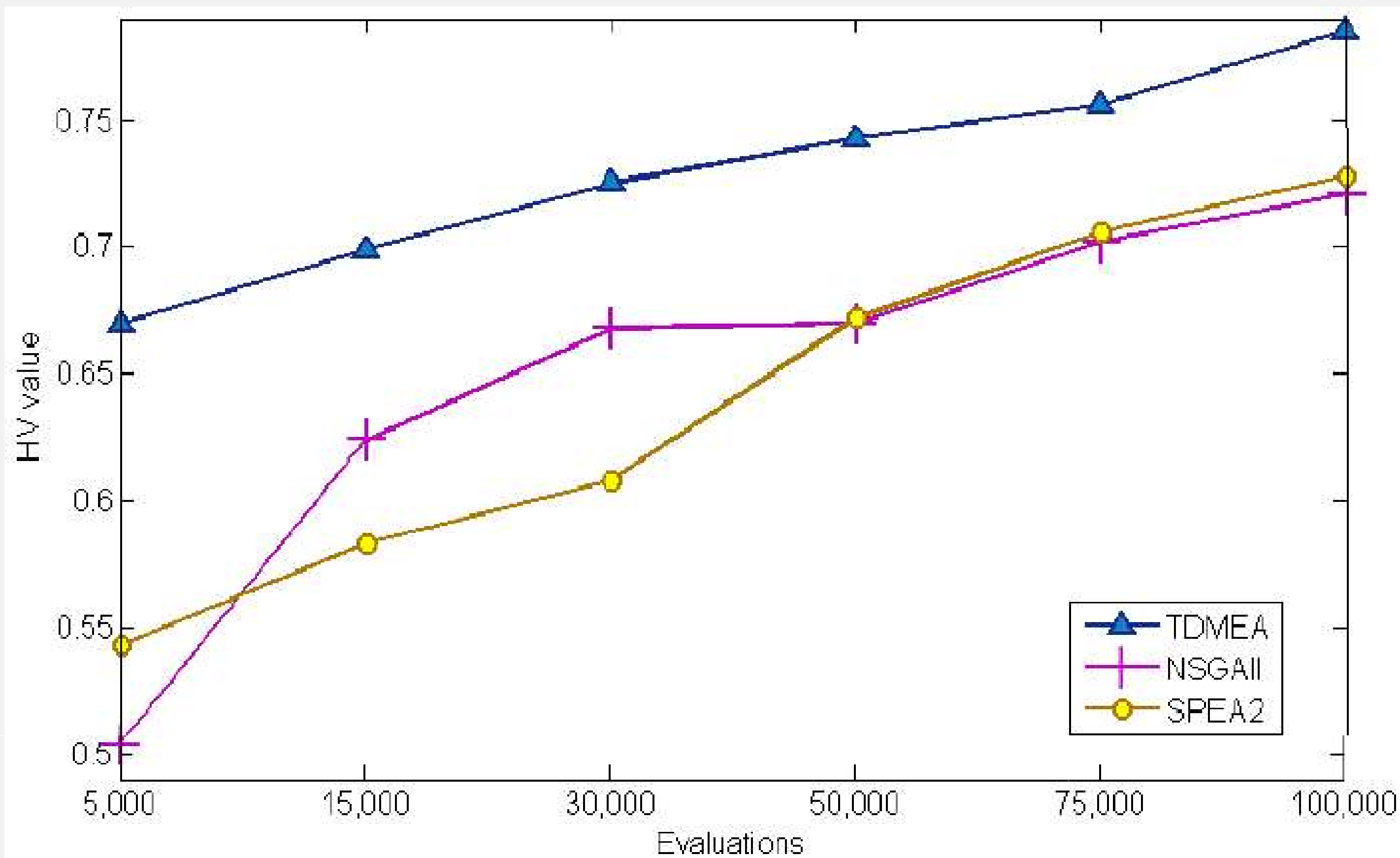
The Mean – Variance efficient frontier for the port7 problem



The Mean – MAD efficient frontier for the SP500 problem



Evolution trace of HV metric for port5 problem



Mean total CPU times (in seconds) for solving the cardinality constrained portfolio optimization problem under different risk measures for 100,000 functions evaluations

Problem	Time in seconds		
	TDMEA	NSGA II	SPEA2
port5 (Mean – Variance)	3.041	167.372	171.409
port6 (Mean – Variance)	3.047	575.877	585.343
port7 (Mean – Variance)	4.922	4573.807	4635.934
FTSE-100 (Mean-Semivariance)	2.897	38.384	40.811
SP500 (Mean – VaR)	2.838	546.725	552.157
SP500 (Mean – CVaR)	2.956	578.541	585.983
SP500 (Mean – MAD)	2.844	539.434	546.293

Mean total CPU times (in seconds) for solving the cardinality constrained portfolio optimization problem under different risk measures for 100,000 functions evaluations