10 Other Topics

10.1 Ranking and Social Choice

Combining feedback from multiple users to rank a collection of items is an important task. We rank movies, restaurants, web pages, and many other items. Ranking has become a multi-billion dollar industry as organizations try to raise the position of their web pages in the results returned by search engines to relevant queries. Developing a method of ranking that cannot be easily gamed by those involved is an important task.

A ranking of a collection of items is defined as a complete ordering. For every pair of items a and b, either a is preferred to b or b is preferred to a. Furthermore, a ranking is transitive in that $a > b$ and $b > c$ implies $a > c$.

One problem of interest in ranking is that of combining many individual rankings into one global ranking. However, merging ranked lists in a meaningful way is non-trivial as the following example illustrates.

Example: Suppose there are three individuals who rank items a, b, and c as illustrated in the following table.

individual	first item	second item	third item
1	a	b	c
2	b	c	a
3	c	a	b

Suppose our algorithm tried to rank the items by first comparing a to b and then comparing b to c. In comparing a to b, two of the three individuals prefer a to b and thus we conclude a is preferable to b. In comparing b to c, again two of the three individuals prefer b to c and we conclude that b is preferable to c. Now by transitivity one would expect that the individuals would prefer a to c, but such is not the case, only one of the individuals prefers a to c and thus c is preferable to a. We come to the illogical conclusion that a is preferable to b, b is preferable to c, and c is preferable to a. ∎

Suppose there are a number of individuals or voters and a set of candidates to be ranked. Each voter produces a ranked list of the candidates. From the set of ranked lists can one construct a reasonable single ranking of the candidates? Assume the method of producing a global ranking is required to satisfy the following three axioms.

Non-dictatorship – The algorithm cannot always simply select one individual's ranking to use as the global ranking.

Unanimity – If every individual prefers a to b, then the global ranking must prefer a to b.

Independent of irrelevant alternatives – If individuals modify their rankings but keep the order of a and b unchanged, then the global order of a and b should not change.

Arrow showed that it is not possible to satisfy all three of the above axioms. We begin with a technical lemma.

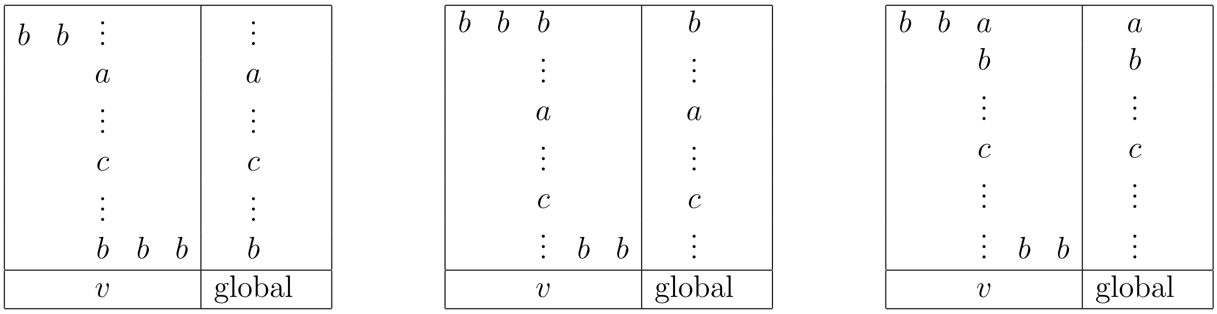
Lemma 10.1 *For a set of rankings in which each individual ranks an item b either first or last (some individuals may rank b first and others may rank b last), a global ranking satisfying the above axioms must put b first or last.*

Proof: Let a, b, and c be distinct items. Suppose to the contrary that b is not first or last in the global ranking. Then there exist a and c where the global ranking puts $a > b$ and $b > c$. By transitivity, the global ranking puts $a > c$. Note that all individuals can move c above a without affecting the order of b and a or the order of b and c since b was first or last on each list. Thus, by independence of irrelevant alternatives, the global ranking would continue to rank $a > b$ and $b > c$ even if all individuals moved c above a since that would not change the individuals relative order of a and b or the individuals relative order of b and c. But then by unanimity, the global ranking would need to put $c > a$, a contradiction. We conclude that the global ranking puts b first or last. ∎

Theorem 10.2 ***(Arrow)*** *Any deterministic algorithm for creating a global ranking from individual rankings of three or more elements in which the global ranking satisfies unanimity and independence of irrelevant alternatives is a dictatorship.*

Proof: Let a, b, and c be distinct items. Consider a set of rankings in which every individual ranks b last. By unanimity, the global ranking must also rank b last. Let the individuals, one by one, move b from bottom to top leaving the other rankings in place. By unanimity, the global ranking must eventually move b from the bottom all the way to the top. When b first moves, it must move all the way to the top by Lemma 10.1.

Let v be the first individual whose change causes the global ranking of b to change. We argue that v is a dictator. First, we argue that v is a dictator for any pair ac not involving b. We will refer to the three rankings of v in Figure 10.1. The first ranking of v is the ranking prior to v moving b from the bottom to the top and the second is the ranking just after v has moved b to the top. Choose any pair ac where a is above c in v's ranking. The third ranking of v is obtained by moving a above b in the second ranking so that $a > b > c$ in v's ranking. By independence of irrelevant alternatives, the global ranking after v has switched to the third ranking puts $a > b$ since all individual ab votes are the same as in the first ranking, where the global ranking placed $a > b$. Similarly $b > c$ in the global ranking since all individual bc votes are the same as in the second ranking, in which b was at the top of the global ranking. By transitivity the global ranking must put $a > c$ and thus the global ranking of a and c agrees with v.



first ranking second ranking third ranking

Figure 10.1: The three rankings that are used in the proof of Theorem 10.2.

Now all individuals except v can modify their rankings arbitrarily while leaving b in its extreme position and by independence of irrelevant alternatives, this does not affect the global ranking of $a > b$ or of $b > c$. Thus, by transitivity this does not affect the global ranking of a and c. Next, all individuals except v can move b to any position without affecting the global ranking of a and c.

At this point we have argued that independent of other individuals' rankings, the global ranking of a and c will agree with v's ranking. Now v can change its ranking arbitrarily, provided it maintains the order of a and c, and by independence of irrelevant alternatives the global ranking of a and c will not change and hence will agree with v. Thus, we conclude that for all a and c, the global ranking agrees with v independent of the other rankings except for the placement of b. But other rankings can move b without changing the global order of other elements. Thus, v is a dictator for the ranking of any pair of elements not involving b.

Note that v changed the relative order of a and b in the global ranking when it moved b from the bottom to the top in the previous argument. We will use this in a moment.

To show that individual v is also a dictator over every pair ab, repeat the construction showing that v is a dictator for every pair ac not involving b only this time place c at the bottom. There must be an individual v_c who is a dictator for any pair such as ab not involving c. Since both v and v_c can affect the global ranking of a and b independent of each other, it must be that v_c is actually v. Thus, the global ranking agrees with v no matter how the other voters modify their rankings. ∎

10.1.1 Randomization

An interesting randomized algorithm that satisfies unanimity and independence of irrelevant alternatives is to pick a random individual and use that individual's ranking as the output. This is called the "random dictator" rule because it is a randomization over dictatorships. An analogous scheme in the context of voting would be to select a winner with probability proportional to the number of votes for that candidate, because this is the same as selecting a random voter and telling that voter to determine the winner. Note that this method has the appealing property that as a voter, there is never any reason to strategize, e.g., voting for candidate a rather than your preferred candidate b because you think b is unlikely to win and you don't want to throw away your vote. With this method, you should always vote for your preferred candidate.

10.1.2 Examples

Borda Count: Suppose we view each individual's ranking as giving each item a score: putting an item in last place gives it one point, putting it in second-to-last place gives it two points, third-to-last place is three points, and so on. In this case, one simple way to combine rankings is to sum up the total number of points received by each item and then sort by total points. This is called the extended Borda Count method.

Let's examine which axioms are satisfied by this approach. It is easy to see that it is a non-dictatorship. It also satisfies unanimity: if every individual prefers a to b, then every individual gives more points to a than to b, and so a will receive a higher total than b. By Arrow's theorem, the approach must fail independence of irrelevant alternatives, and indeed this is the case. Here is a simple example with three voters and four items $\{a, b, c, d\}$ where the independence of irrelevant alternatives axiom fails:

individual	ranking
1	$abcd$
2	$abcd$
3	$bacd$

In this example, a receives 11 points and is ranked first, b receives 10 points and is ranked second, c receives 6 points and is ranked third, and d receives 3 points and is ranked fourth. However, if individual 3 changes his ranking to $bcda$, then this reduces the total number of points received by a to 9, and so b is now ranked first overall. Thus, even though individual 3's relative order of b and a did not change, and indeed no individual's relative order of b and a changed, the global order of b and a did change.

Hare voting: An interesting system for voting is to have everyone vote for their favorite candidate. If some candidate receives a majority of the votes, he or she is declared the winner. If no candidate receives a majority of votes, the candidate with the fewest votes is dropped from the slate and the process is repeated.

The Hare system implements this method by asking each voter to rank all the candidates. Then one counts how many voters ranked each candidate as number one. If no candidate receives a majority, the candidate with the fewest number one votes is dropped from each voters ranking. If the dropped candidate was number one on some voters list, then the number two candidate becomes that voter's number one choice. The process of counting the number one rankings is then repeated.

We can convert the Hare voting system into a ranking method in the following way. Whichever candidate is dropped first is put in last place, whichever is dropped second is put in second-to-last place, and so on, until the system selects a winner, which is put in first place. The candidates remaining, if any, are placed between the first-place candidate and the candidates who were dropped, in an order determined by running this procedure recursively on just those remaining candidates.

As with Borda Count, the Hare system also fails to satisfy independence of irrelevant alternatives. Consider the following situation in which there are 21 voters that fall into four categories. Voters within a category rank individuals in the same order.

Category	Number of voters in category	Preference order
1	7	abcd
2	6	bacd
3	5	cbad
4	3	dcba

The Hare system would first eliminate d since d gets only three rank one votes. Then it would eliminate b since b gets only six rank one votes whereas a gets seven and c gets eight. At this point a is declared the winner since a has thirteen votes to c's eight votes. So, the final ranking is *acbd*.

Now assume that Category 4 voters who prefer b to a move b up to first place. This keeps their order of a and b unchanged, but it reverses the global order of a and b. In particular, d is first eliminated since it gets no rank one votes. Then c with five votes is eliminated. Finally, b is declared the winner with 14 votes, so the final ranking is *bacd*.

Interestingly, Category 4 voters who dislike a and have ranked a last could prevent a from winning by moving a up to first. Ironically this results in eliminating d, then c, with five votes and declaring b the winner with 11 votes. Note that by moving a up, category 4 voters were able to deny a the election and get b to win, whom they prefer over a.

10.2 Compressed Sensing and Sparse Vectors

Define a *signal* to be a vector $\mathbf{x}$ of length d, and define a *measurement* of $\mathbf{x}$ to be a dot-product of $\mathbf{x}$ with some known vector $\mathbf{a_i}$. If we wish to uniquely reconstruct $\mathbf{x}$ without any assumptions, then d linearly-independent measurements are necessary and sufficient.