Σήμερα: Ασκήσεις στις Βασικές Έννοιες

Άσκηση 1

Στο σύνολο $\mathbb{N}$ ορίζουμε την εξής σχέση:

$$x R y \iff \text{Υπάρχει } k \in \mathbb{Z} \text{ ώστε } y - x = 3k$$

α) Να δειχθεί ότι η σχέση R είναι σχέση ισοδυναμίας στο $\mathbb{N}$

Παραδείγματα

$2R5$ διότι $5-2 = 3 \cdot \underset{k=1}{1}$

$2R8$ διότι $8-2 = 3 \cdot \underset{k=2}{2}$

$8R2$ διότι $2-8 = 3 \cdot \underset{k=-2}{(-2)}$

$3 \not R 7$ διότι δεν υπάρχει $k \in \mathbb{Z}$ ώστε
$7-3 = 3k \iff 4 = 3k$

Υπενθύμιση: Μια σχέση R στο E ονομάζεται σχέση ισοδυναμίας ανν

- $\alpha R \alpha$ για κάθε $\alpha \in E$ (ανακλαστική)
- Αν $\alpha R b$ τότε $b R \alpha$ (και αντίστροφα) $\forall \alpha, b \in E$ (συμμετρική)
- Αν $\alpha R b$ και $b R c$, τότε $\alpha R c$ (μεταβατική) $\forall \alpha, b, c \in E$

• Για καθε $x \in \mathbb{N}$ ισχυει

$$\underset{k=0}{x - x = 3 \cdot 0} \Rightarrow xRx$$

Άρα, ισχυει η ανακλαστική ιδιότητα

• Εσπω $x, y \in \mathbb{N}$ με xRy τότε

υπάρχει $k \in \mathbb{Z}$ ώστε

$$y - x = 3k \Leftrightarrow$$
$$x - y = 3(-k)$$

Επειδη $-k \in \mathbb{Z}$ έπεται οτι yRx

Άρα, ισχύει η συμμετρική ιδιοτητα

• Εστω $x, y, z \in \mathbb{N}$ με xRy και yRz τότε

υπαρχει $k_1 \in \mathbb{Z}$ ώστε $y - x = 3k_1$

υπάρχει $k_2 \in \mathbb{Z}$ ώστε $z - y = 3k_2$

Προσθετοντας $z - x = 3(k_1 + k_2)$

Επειδη $k_1 + k_2 \in \mathbb{Z}$ επεται οτι xRz

Αρα, ισχυει η μεταβατική ιδιότητα

Άρα R σχεση ισοδυναμιας στο $\mathbb{N}$

β) Να βρεθει η κλάση ισοδυναμιας του αριθμου 2.

Ψαχνουμε ολα τα $y \in \mathbb{N}$ ωστε

$2Ry \Leftrightarrow$ Υπάρχει $k \in \mathbb{Z}$ ωστε

$$y - 2 = 3k \Leftrightarrow$$

$$\boxed{y = 3k + 2,\ k \in \mathbb{Z}}$$

Για $k=0$ $y=2$
$k=1$ $y=5$
$k=2$ $y=8$
$k=3$ $y=11$
$k=4$ $y=14$

Αρα, η κλάση του 2 ειναι το συνολο

$$C_2 = \{2, 5, 8, 11, 14, \ldots\}$$

γ) Να βρεθεί το σύνολο πηλίκο της σχέσης R

Διαλέγουμε εναν αριθμό που δεν ανήκει στο C_2
Π.χ. 1
$C_1 = \{y \in \mathbb{N} : 1Ry\}$

$1Ry \Leftrightarrow y-1=3k$ για κάποιο $k \in \mathbb{Z}$

$\Leftrightarrow y=3k+1$

$C_1 = \{\underset{k=0}{1}, \underset{k=1}{4}, \underset{k=2}{7}, 10, \underset{k=4}{13}, 16, \dots\}$

Επειδή $C_1 \cup C_2 \neq \mathbb{N}$ υπάρχει και άλλη κλάση στην σχέση R

Διαλέγουμε εναν αριθμο που δεν ανήκει στο $C_1 \cup C_2$

Πχ. 3

$C_3 = \{y \in \mathbb{N} : 3Ry\} = \{0, 3, 6, 9, 12, \dots\}$

Επειδη $C_1 \cup C_2 \cup C_3 = \mathbb{N}$ επεται ότι

Βρήκαμε ολες της κλάσεις της R

Άρα, το συνολο πηλίκο αποτελειται απο τα σύνολα C_1, C_2, C_3.

$R/_{\sim} = \{C_1, C_2, C_3\}$

Άσκηση 2

Στο σύνολο $\mathbb{N}^*$ ορίζουμε την σχέση διαιρετότητας ως εξής

$$x \mid y \iff \text{ο } x \text{ διαιρεί τον } y \text{, ή ισοδύναμα}$$
$$\text{υπάρχει } k \in \mathbb{N}^* \text{ ώστε } y = kx$$

α) Να δειχθεί ότι η σχέση $\mid$ είναι σχέση μερικής διάταξης.

<u>Υπενθύμιση</u>: R στο E <u>σχέση μερικής διάταξης</u>

- aRa για κάθε $a \in E$
- Αν aRb και bRa, τότε $a=b$ $\forall a,b \in E$ (αντισυμμετρική)
- Αν aRb και bRc, τότε aRc $\forall a,b,c \in E$

Για κάθε $x \in \mathbb{N}^*$

$x = 1 \cdot x$, άρα $x \mid x$.

Για κάθε $x, y \in \mathbb{N}^*$ με

$x \mid y$ και $y \mid x$ υπάρχουν $\boxed{k_1, k_2 \in \mathbb{N}^*}$ ώστε

$$y = k_1 x \text{ και } x = k_2 y \Rightarrow y = k_1 k_2 y$$

$$\iff k_1 k_2 = 1 \iff k_1 = k_2 = 1 \Rightarrow x = y.$$

Για κάθε $x, y, z \in \mathbb{N}^*$ με
$x|y$ και $y|z$ ισχύει ότι υπάρχουν
$k_1, k_2 \in \mathbb{N}^*$ ώστε

$$y = k_1 x \text{ και } z = k_2 y$$

Άρα $$z = k_1 k_2 x$$

Επειδή $k_1 k_2 \in \mathbb{N}^*$ έπεται ότι $x|z$

Άρα, η σχέση διαιρετότητας είναι μερική διάταξη στο $\mathbb{N}^*$.

β) Είναι η σχέση διαιρετότητας ολική διάταξη στο $\mathbb{N}^*$;

Υπενθύμιση R ολική διάταξη
Ισχύει επιπλέον ότι για κάθε $a, b \in E$
είτε aRb είτε bRa είτε και τα δύο

Δεν είναι ολική διάταξη διότι

$$3 \nmid 5 \text{ και } 5 \nmid 3$$

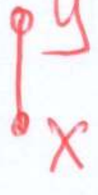
Έστω A το σύνολο των θετικών διαιρετών του $24 = 3 \cdot 8$

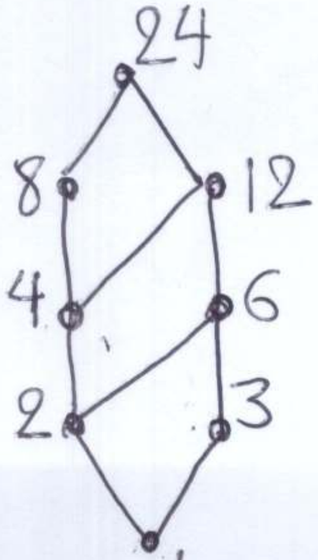
$A = \{1, 2, 3, 4, 6, 8, 12, 24\}$

Θεωρούμε το σύνολο A εφοδιασμένο με την μερική διάταξη της διαιρετότητας $|$

Μπορούμε να αναπαραστήσουμε τις συγκρίσεις των στοιχείων του A με βάση την σχέση διαιρετότητας $|$ χρησιμοποιώντας το διάγραμμα Hasse

- Τα στοιχεία είναι σημεία
- Αν $x \lessdot y$ τότε





Διάγραμμα Hasse της σχέσης $|$

Να βρεθει το supremum και το infimum του A

Για το supremum του A
Ψαχνουμε φυσικο αριθμο x ωστε

$1|x,\ 2|x,\ 3|x, \ldots, 24|x$

και το x να ειναι το ελαχιστο δυνατον

supA = εκπ των στοιχειων του = 24
infA = μκδ των στοιχειων του = 1

Να βρεθει το supB και infB οπου

$B=\{2,4,3\}$

$\sup B = \text{εκπ}(2,4,3) = 12$
$\inf B = \text{μκδ}(2,4,3) = 1$

Σελ. 38, Διαλεξεις 4, 5, 6

- $A_1 = \{ \frac{1}{3^n} : n \in \mathbb{N} \}$

$\sup A_1 = 1$ (προκυπτει για $n=0$)
και ειναι και το μεγιστο του A_1
$\inf A_1 = 0$
και το A_1 δεν εχει ελαχιστο

- $A_3 = \{ \sum_{k=1}^{n} \frac{1}{2^k} : n \in \mathbb{N}^* \}$

Υπενθύμιση: $\sum_{k=1}^{n} x^k = x + x^2 + x^3 + \dots + x^n = \frac{x^{n+1} - x}{x - 1}$

$\alpha_n = \sum_{k=1}^{n} \frac{1}{2^k}$

$\alpha_1 = \frac{1}{2}$, $\alpha_2 = \frac{1}{2} + \frac{1}{2^2}$, $\alpha_3 = \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3}$

$\alpha_1 < \alpha_2 < \alpha_3 < \dots$

Αρα $\inf A_3 = \min A_3 = \alpha_1 = \frac{1}{2}$

Για το <u>supremum</u> παρατηρουμε $\sum_{k=1}^{n} \frac{1}{2^k} < 1$

$$\alpha_n = \sum_{k=1}^{n} \frac{1}{2^k} = \frac{\left(\frac{1}{2}\right)^{n+1} - \frac{1}{2}}{\frac{1}{2} - 1} = \frac{\frac{1}{2} - \left(\frac{1}{2}\right)^{n+1}}{\frac{1}{2}} = 1 - \left(\frac{1}{2}\right)^n$$

Προφανώς $\alpha_n = 1 - \left(\frac{1}{2}\right)^n < 1$

Άρα, το 1 είναι άνω φράγμα του A_3.

Θα δείξουμε ότι για κάθε αριθμό $\alpha < 1$ υπάρχει n ώστε $\alpha_n > \alpha$

Θέτουμε $\alpha = 1 - \varepsilon$ οπου $\varepsilon > 0$ και θα αποδείξουμε με άτοπο ότι δεν ισχυει η ανισότητα $\alpha_n \leq 1 - \varepsilon$ για κάθε $n \in \mathbb{N}^*$

Πράγματι

$$\alpha_n \leq 1 - \varepsilon \iff 1 - \left(\frac{1}{2}\right)^n \leq 1 - \varepsilon \iff$$

$$\varepsilon \leq \left(\frac{1}{2}\right)^n \iff 2^n \leq \frac{1}{\varepsilon} \text{ για κάθε } n \in \mathbb{N}^*$$

άτοπο

Άρα, δεν υπαρχει $\alpha < 1$ ώστε $\alpha_n < \alpha \ \forall n \in \mathbb{N}^*$

Επομένως $\sup A_3 = 1$. Το A_3 δεν εχει maximum

Άσκηση 3 (Διώνυμο του Νευτωνα)

Υπενθυμιση $\quad \alpha, b \in \mathbb{R},\ n \in \mathbb{N}^*$

$$\boxed{(\alpha+b)^n = \sum_{k=0}^{n} \binom{n}{k} \alpha^k b^{n-k}}$$

$$(b+\alpha)^n = \sum_{k=0}^{n} \binom{n}{k} \alpha^{n-k} b^k$$

α) Να βρεθει ο συντελεστης του α^5 αναπτυγμα της εκφρασης $\left(\alpha + \frac{1}{\alpha^2}\right)^9$

Απο τον τυπο του Νευτωνα

$$\left(\alpha+\frac{1}{\alpha^2}\right)^9 = \sum_{k=0}^{9} \binom{9}{k} \alpha^k \left(\frac{1}{\alpha^2}\right)^{9-k}$$

$$= \sum_{k=0}^{9} \binom{9}{k} \alpha^{\underline{-18+3k}}$$

Πρεπει $-18+3k=5 \Leftrightarrow 3k=23$ αδυνατο

Άρα, το αθροισμα δεν περιεχει τον ορο α^5

Δηλαδη ο συντελεστης του ειναι 0

β) Το ιδιο ερωτημα για τον συντελεστη του α^3

Πρέπει $-18+3k=3 \Leftrightarrow 3k=21$

$\Leftrightarrow k=7$

Άρα ο συντελεστης του α^3 ειναι $\binom{9}{7}$

γ) Να βρεθει η τιμη των παρακάτω αθροισματω

$$S_n=\sum_{k=0}^{n}\binom{n}{k}$$

$$\sum_{k=0}^{n}\binom{n}{k}=\sum_{k=0}^{n}\binom{n}{k}1^{k}\cdot 1^{n-k}$$

$$=(1+1)^n=2^n$$

$$S_n=\sum_{k=0}^{n}2^{k}\binom{n}{k}$$

$$\sum_{k=0}^{n}2^{k}\binom{n}{k}=\sum_{k=0}^{n}2^{k}\cdot 1^{n-k}\binom{n}{k}$$

$$=(2+1)^n=3^n$$

$$S_n = \sum_{k=0}^{n} 2^n \binom{n}{k}$$

$$\sum_{k=0}^{n} 2^n \binom{n}{k} = 2^n \sum_{k=0}^{n} \binom{n}{k} = 2^n \cdot 2^n$$

$$= 2^{2n} = 4^n$$

$$S_n = \sum_{k=0}^{n} n \binom{n}{k}$$

$$\sum_{k=0}^{n} n \binom{n}{k} = n \sum_{k=0}^{n} \binom{n}{k} = n\, 2^n$$

$$S_n = \sum_{k=0}^{n} k \binom{n}{k}$$

$$\binom{n}{k} = \frac{n}{k} \binom{n-1}{k-1} \Leftrightarrow k \binom{n}{k} = n \binom{n-1}{k-1}$$

$$\sum_{k=0}^{n} k \binom{n}{k} = \sum_{k=1}^{n} k \binom{n}{k} = \sum_{k=1}^{n} n \binom{n-1}{k-1}$$

$$= n \sum_{k=1}^{n} \binom{n-1}{k-1} = n \sum_{\lambda=0}^{n-1} \binom{n-1}{\lambda} = n\, 2^{n-1}$$

$$k-1 = \lambda$$

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1.8 Παράρτημα: Το παράδοξο του Russell

Η μέθοδος ορισμού συνόλων με τη βοήθεια μιας *οποιασδήποτε ιδιότητας*, π.χ.

$$A = \{x \in \mathbb{N} : x \text{ είναι άρτιος}\},$$

$$B = \{\text{Το σύνολο των φοιτητών του Τμήματος Πληροφορικής}\},$$

$$C = \{\text{Το σύνολο των ψηλών ανθρώπων}\},$$

μπορεί να οδηγήσει σε παράδοξα και γενικά επιτρέπεται **μόνο** αν υπάρχει ένα σύνολο αναφοράς από το οποίο επιλέγονται τα στοιχεία αυτών των συνόλων, και μόνο αν η ιδιότητα τηρεί ορισμένες προϋποθέσεις. (Για το A το σύνολο αναφοράς είναι το $\mathbb{N}$, ενώ για το B το σύνολο αναφοράς είναι το σύνολο όλων των φοιτητών, ενώ το σύνολο C δεν ορίζεται διότι η ιδιότητα ορισμού του δεν είναι σαφώς καθορισμένη.)

Πιο συγκεκριμένα **δεν ορίζεται** *το σύνολο όλων των συνόλων*. Πράγματι, η παραδοχή της ύπαρξης του συνόλου όλων των συνόλων οδηγεί στο επόμενο παράδοξο, το οποίο οφείλεται στον Bertrand Russell (1872–1970).

Αν δεχτούμε ότι υπάρχει το σύνολο όλων των συνόλων τότε μπορούμε να ορίσουμε σύνολα όπως το επόμενο:

$$P = \{\text{σύνολο } X : X \notin X\},$$

δηλαδή το P είναι το σύνολο όλων συνόλων X με την ιδιότητα $X \notin X$. (Εδώ το σύνολο αναφοράς είναι το σύνολο όλων των συνόλων από το οποίο επιλέγονται τα σύνολα με την ιδιότητα $X \notin X$).

Το παράδοξο προκύπτει αν προσπαθήσουμε να ελέγξουμε αν ισχύει $P \in P$ ή $P \notin P$.

Αν $P \in P$, τότε πρέπει $P \notin P$.

Αν $P \notin P$, τότε πρέπει $P \in P$.

Επομένως, και στις δύο περιπτώσεις έχουμε αντίφαση. Οδηγηθήκαμε σε αντίφαση διότι θεωρήσαμε ότι υπάρχει ως σύνολο αναφοράς το σύνολο όλων των συνόλων.

Η παραπάνω κατασκευή ονομάζεται παράδοξο του Russell.

Περισσότερα στοιχεία για τα προβλήματα της θεμελίωσης των συνόλων περιέχονται στα κεφαλαία 1 και 3 του βιβλίου *Σημειώσεις στη συνολοθεωρία* του Γιάννη Ν. Μοσχοβάκη, το οποίο είναι διαθέσιμο και από το σύνδεσμο: http://www.math.ucla.edu/~ynm/lectures/g.pdf

Letter to Frege

BERTRAND RUSSELL

(*1902*)

Bertrand Russell discovered what became known as the Russell paradox in June 1901 (see *1944*, p. 13). In the letter below, written more than a year later and hitherto unpublished, he communicates the paradox to Frege. The paradox shook the logicians' world, and the rumbles are still felt today.

The Burali-Forti paradox, discovered a few years earlier, involves the notion of ordinal number; it seemed to be intimately connected with Cantor's set theory, hence to be the mathematicians' concern rather than the logicians'. Russell's paradox, which makes use of the bare notions of set and element, falls squarely in the field of logic. The paradox was first published by Russell in *The principles of mathematics* (*1903*) and is discussed there in great detail (see especially pp. 101–107). After various attempts, Russell considered the paradox solved by the theory of types (*1908a*). Zermelo (below, p. 191, footnote 9) states that he had discovered the paradox independently of Russell and communicated it to Hilbert, among others, prior to its publication by Russell.

In addition to the statement of the paradox, the letter offers a vivid picture of Russell's attitude toward Frege and his work at the time.

The formula in Peano's notation at the end of the letter can be read more easily if one compares it with formula 450 in *Peano 1898a*, p. VII (or *1897*, p. 15).

Russell wrote the letter in German, and it was translated by Beverly Woodward. Lord Russell read the translation and gave permission to print it here.

Friday's Hill, Haslemere, 16 June 1902

Dear colleague,

For a year and a half I have been acquainted with your *Grundgesetze der Arithmetik*, but it is only now that I have been able to find the time for the thorough study I intended to make of your work. I find myself in complete agreement with you in all essentials, particularly when you reject any psychological element ⟦Moment⟧ in logic and when you place a high value upon an ideography ⟦Begriffsschrift⟧ for the foundations of mathematics and of formal logic, which, incidentally, can hardly be distinguished. With regard to many particular questions, I find in your work discussions, distinctions, and definitions that one seeks in vain in the works of other logicians. Especially so far as function is concerned (§ 9 of your *Begriffsschrift*), I have been led on my own to views that are the same even in the details. There is just one point where I have encountered a difficulty. You state (p. 17 ⟦p. 23 above⟧) that a function,

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too, can act as the indeterminate element. This I formerly believed, but now this view seems doubtful to me because of the following contradiction. Let w be the predicate: to be a predicate that cannot be predicated of itself. Can w be predicated of itself? From each answer its opposite follows. Therefore we must conclude that w is not a predicate. Likewise there is no class (as a totality) of those classes which, each taken as a totality, do not belong to themselves. From this I conclude that under certain circumstances a definable collection ⟦Menge⟧ does not form a totality.

I am on the point of finishing a book on the principles of mathematics and in it I should like to discuss your work very thoroughly.[1] I already have your books or shall buy them soon, but I would be very grateful to you if you could send me reprints of your articles in various periodicals. In case this should be impossible, however, I will obtain them from a library.

The exact treatment of logic in fundamental questions, where symbols fail, has remained very much behind; in your works I find the best I know of our time, and therefore I have permitted myself to express my deep respect to you. It is very regrettable that you have not come to publish the second volume of your *Grundgesetze*; I hope that this will still be done.

Very respectfully yours,

BERTRAND RUSSELL

The above contradiction, when expressed in Peano's ideography, reads as follows:

$$w = \mathrm{cls} \cap x \ni (x \sim\varepsilon\, x) . \supset : w\, \varepsilon\, w . = . w \sim\varepsilon\, w.$$

I have written to Peano about this, but he still owes me an answer.

[1] ⟦This was done in *Russell 1903*, Appendix A, "The logical and arithmetical doctrines of Frege".⟧

Letter to Russell

GOTTLOB FREGE

(*1902*)

This is Frege's prompt answer to Russell's letter published above. Frege first calls Russell's attention to an error in *Begriffsschrift*; it is a mere oversight, without any consequence (see above, p. 15, footnote 12). He then describes his reaction to the paradox that Russell has just communicated to him, and he begins to look for the source of the predicament. He incriminates the "transformation of the generalization of an equality into an equality of courses-of-values". For Frege a function is something incomplete, "unsaturated". When it is written $f(x)$, x is something extraneous that merely serves to indicate the kind of supplementation that is needed; we might just as well write $f(\ )$. Consider now two functions that, for the same argument, always have the same value: $(x)(f(x) = g(x))$. (This is not Frege's notation, but its modern equivalent.) Since f and g, or rather $f(\ )$ and $g(\ )$, are something incomplete, we cannot simply write $f = g$. Functions are not objects, and in order to treat them, in some respect, as objects Frege introduces their *Werthverlauf*. The *Werthverlauf* of a function $f(x)$ is denoted by $\acute{\varepsilon}f(\varepsilon)$ (where ε is a dummy; we can also write $\acute{\alpha}f(\alpha), \ldots$). The expression "the function $f(x)$ has the same *Werthverlauf* as the function $g(x)$" is taken to mean "for the same argument the function $f(x)$ always has the same value as the function $g(x)$", and we can write (in modern notation)

$$(*) \quad (x)(f(x) = g(x)) \equiv (\acute{\varepsilon}f(\varepsilon) = \acute{\alpha}g(\alpha)).$$

This is the "transformation of the generalization of an equality into an equality of courses-of-values". Whereas the function is unsaturated and is not an object, its *Werthverlauf* is "something complete in itself", an object, in particular so far as substitution is concerned. There Frege sees the origin of the paradox.

Frege soon made his point more specific. He received Russell's letter while the second volume of his *Grundgesetze der Arithmetik* was at the printshop, and he barely had the time to add an appendix in which he shows how the schema (*) above (or rather half of it, the implication from right to left) allows the derivation of the paradox; he also proposed a restriction in the schema to prevent that. Russell, whose *Principles of mathematics* was at the printshop when he received Frege's volume, added to his book an appendix in which he endorsed Frege's emendation. But soon thereafter he tried out various other solutions (*1905a*); he finally proposed his theory of types (*1908a*).

Russell's paradox has been leaven in modern logic, and countless works have dealt with it. For a late and thorough study of Frege's "way out", see *Quine 1955*.

When Lord Russell was asked whether he would consent to the publication of his letter to Frege (*1902*), he replied with the following letter, in which the reader will find a stirring tribute to Frege.

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Penrhyndeudraeth, 23 November 1962

Dear Professor van Heijenoort,

I should be most pleased if you would publish the correspondence between Frege and myself, and I am grateful to you for suggesting this. As I think about acts of integrity and grace, I realise that there is nothing in my knowledge to compare with Frege's dedication to truth. His entire life's work was on the verge of completion, much of his work had been ignored to the benefit of men infinitely less capable, his second volume was about to be published, and upon finding that his fundamental assumption was in error, he responded with intellectual pleasure clearly submerging any feelings of personal disappointment. It was almost superhuman and a telling indication of that of which men are capable if their dedication is to creative work and knowledge instead of cruder efforts to dominate and be known.

Yours sincerely,

Bertrand Russell

The translation of Frege's letter is by Beverly Woodward, and it is printed here with the kind permission of Verlag Felix Meiner and the Institut für mathematische Logik und Grundlagenforschung in Münster, who are preparing an edition of Frege's scientific correspondence and hitherto unpublished writings; this edition will include the German text of the letter.

Jena, 22 June 1902

Dear colleague,

Many thanks for your interesting letter of 16 June. I am pleased that you agree with me on many points and that you intend to discuss my work thoroughly. In response to your request I am sending you the following publications:

1. "Kritische Beleuchtung" ⟦*1895*⟧,
2. "Ueber die Begriffsschrift des Herrn Peano" ⟦*1896*⟧,
3. "Ueber Begriff und Gegenstand" ⟦*1892*⟧,
4. "Über Sinn und Bedeutung" ⟦*1892a*⟧,
5. "Ueber formale Theorien der Arithmetik" ⟦*1885*⟧.

I received an empty envelope that seems to be addressed by your hand. I surmise that you meant to send me something that has been lost by accident. If this is the case, I thank you for your kind intention. I am enclosing the front of the envelope.

When I now read my *Begriffsschrift* again, I find that I have changed my views on many points, as you will see if you compare it with my *Grundgesetze der Arithmetik*. I ask you to delete the paragraph beginning "Nicht minder erkennt man" on page 7 of my *Begriffsschrift* ⟦"It is no less easy to see", p. 15 above⟧, since it is incorrect; incidentally, this had no detrimental effects on the rest of the booklet's contents.

Your discovery of the contradiction caused me the greatest surprise and, I would almost say, consternation, since it has shaken the basis on which I intended to build arithmetic. It seems, then, that transforming the generalization of an equality into an equality of courses-of-values ⟦die Umwandlung der Allgemeinheit einer Gleichheit in eine Werthverlaufsgleichheit⟧ (§ 9 of my *Grundgesetze*) is not always permitted, that my Rule V (§ 20, p. 36) is false, and that my explanations in § 31 are not sufficient to ensure that my combinations of signs have a meaning in all cases. I must reflect further on the matter. It is all the more serious since, with the loss of my Rule V, not

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only the foundations of my arithmetic, but also the sole possible foundations of arithmetic, seem to vanish. Yet, I should think, it must be possible to set up conditions for the transformation of the generalization of an equality into an equality of courses-of-values such that the essentials of my proofs remain intact. In any case your discovery is very remarkable and will perhaps result in a great advance in logic, unwelcome as it may seem at first glance.

Incidentally, it seems to me that the expression "a predicate is predicated of itself" is not exact. A predicate is as a rule a first-level function, and this function requires an object as argument and cannot have itself as argument (subject). Therefore I would prefer to say "a notion is predicated of its own extension". If the function $\Phi(\xi)$ is a concept, I denote its extension (or the corresponding class) by "$\acute{\varepsilon}\Phi(\varepsilon)$" (to be sure, the justification for this has now become questionable to me). In "$\Phi(\acute{\varepsilon}\Phi(\varepsilon))$" or "$\acute{\varepsilon}\Phi(\varepsilon) \cap \acute{\varepsilon}\Phi(\varepsilon)$"[1] we then have a case in which the concept $\Phi(\xi)$ is predicated of its own extension.

The second volume of my *Grundgesetze* is to appear shortly. I shall no doubt have to add an appendix in which your discovery is taken into account. If only I already had the right point of view for that!

Very respectfully yours,

G. Frege

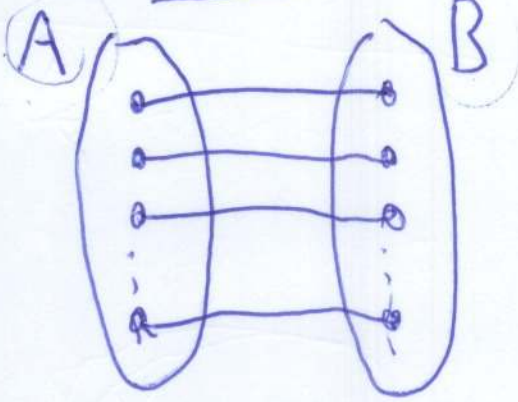
[1] ⟦"$\cap$" is a sign used by Frege for reducing second-level functions to first-level functions. See *Frege 1893*, § 34.⟧

Ισοδύναμα σύνολα – Αριθμήσιμα σύνολα

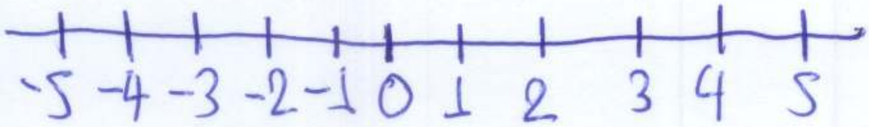
A, B σύνολα

Πότε τα A, B είναι ισοδύναμα;

Αν υπάρχει αμφιμονοσήμαντη απεικόνιση $f: A \to B$



$\mathbb{N}^*$ ισοδύναμο με το $\mathbb{Z}$



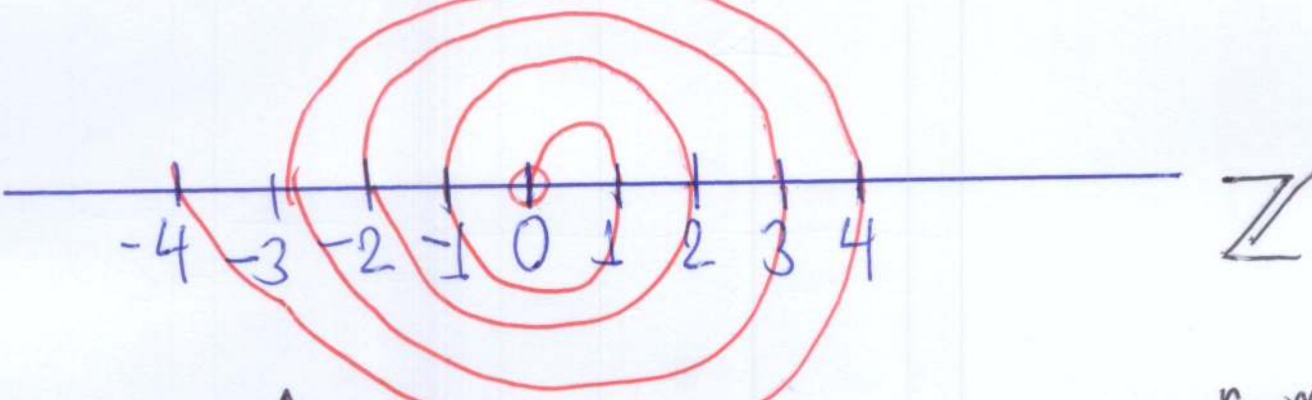
$\mathbb{N}^*$ ισοδύναμο με $\mathbb{N}^* \times \mathbb{N}^*$

Cantor

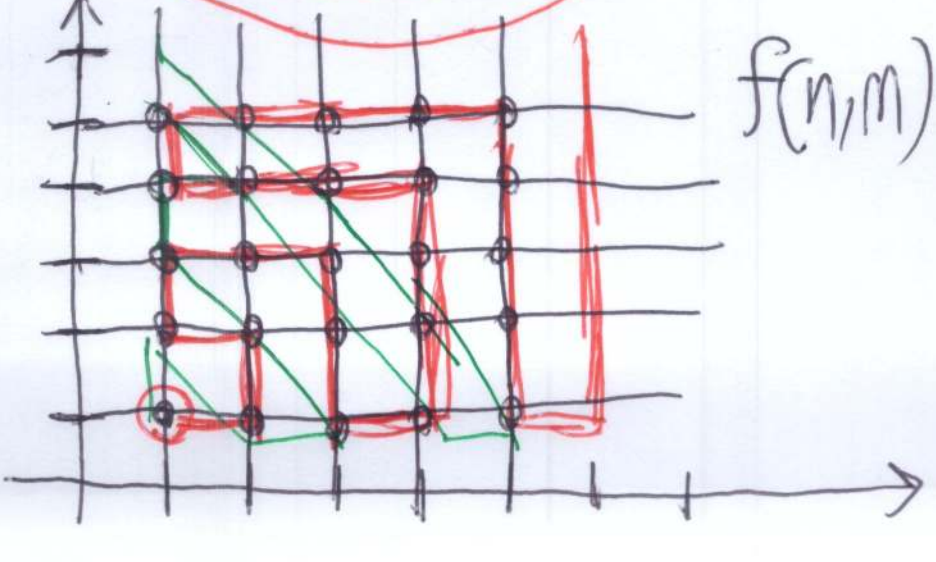
$\mathbb{N}^*$ δεν είναι ισοδύναμο με το $\mathbb{R}$

Τι σημαίνει οτι κάποιο σύνολο A είναι ισοδύναμο με το $\mathbb{N}^*$ δηλαδή είναι αριθμήσιμο;

Μπορούμε να βάλουμε τα στοιχεια του A σε μια σειρα (όπως μπαίνουν οι φυσικοί αριθμοι) και καθε στοιχειο θα εμφανίζεται σε πεπερασμένα βηματα στην σειρα αυτη.

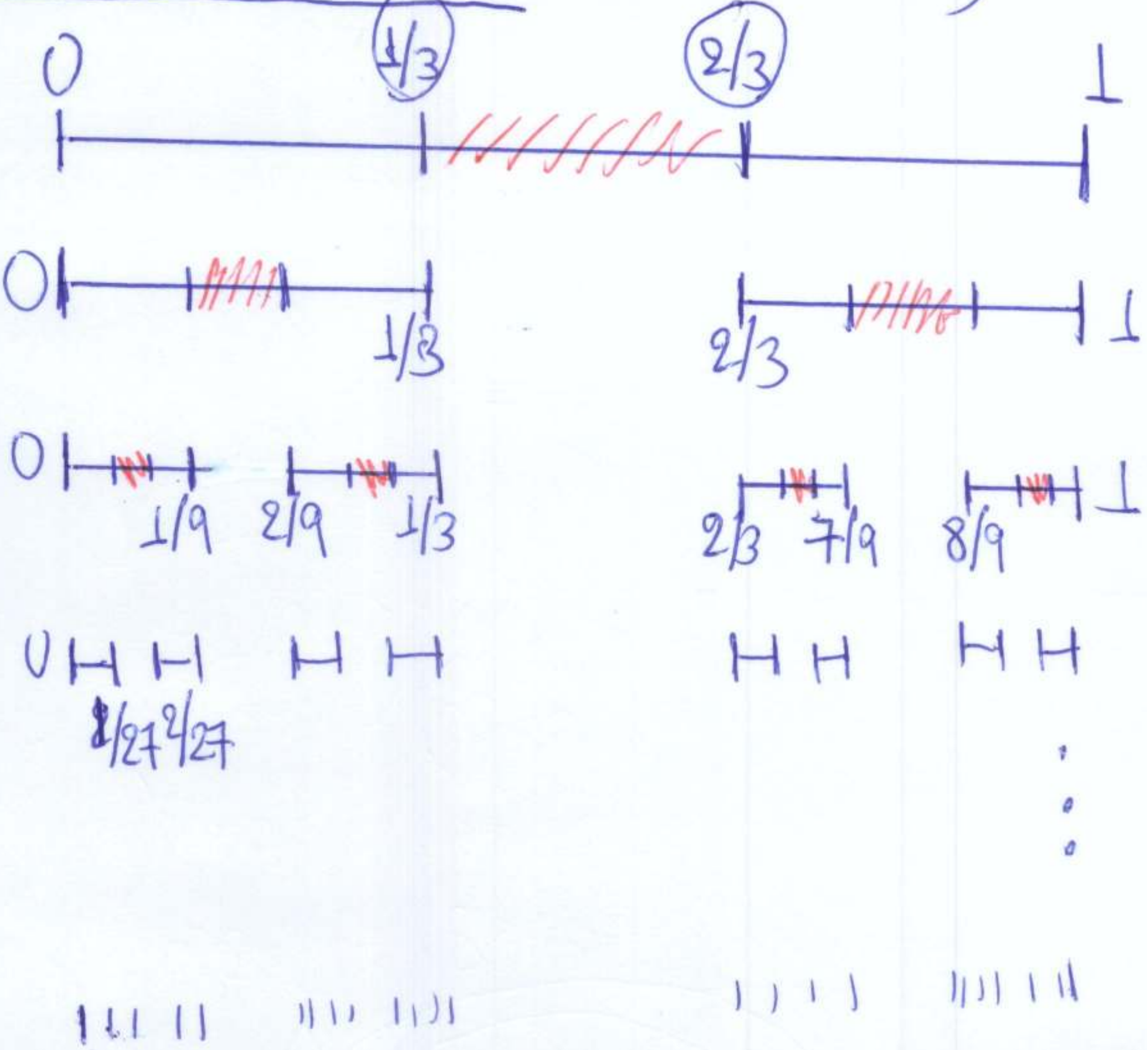


$\mathbb{N}^* \times \mathbb{N}^*$



$f(n,m) = 2^n \cdot 3^m$

Σκόνη του Cantor (fractal)



Το όριο αυτής της διαδικασίας ονομάζεται σύνολο του Cantor ή σκόνη του Cantor

$$1 - \frac{1}{3^1} - \frac{2}{3^2} - \frac{2^2}{3^3} - \frac{2^3}{3^4} - \dots = 1 - 1 = 0$$

Στο τέλος το σύνολο δεν θα περιέχει κανένα <u>διάστημα</u>

Θα περιέχει άπειρα στοιχεία.
Δεν είναι αριθμήσιμο αλλά ισοδύναμο με το $\mathbb{R}$

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WIKIPEDIA

Cantor set

In mathematics, the **Cantor set** is a set of points lying on a single line segment that has a number of remarkable and deep properties. It was discovered in 1874 by Henry John Stephen Smith[1][2][3][4] and introduced by German mathematician Georg Cantor in 1883.[5][6]

Through consideration of this set, Cantor and others helped lay the foundations of modern point-set topology. Although Cantor himself defined the set in a general, abstract way, the most common modern construction is the **Cantor ternary set**, built by removing the middle third of a line segment and then repeating the process with the remaining shorter segments. Cantor himself mentioned the ternary construction only in passing, as an example of a more general idea, that of a perfect set that is nowhere dense.

Zoom in Cantor set. Each point in the set is represented here by a vertical line.

Contents

Construction and formula of the ternary set

The Cantor ternary set $\mathcal{C}$ is created by iteratively deleting the *open* middle third from a set of line segments. One starts by deleting the open middle third $(\frac{1}{3}, \frac{2}{3})$ from the interval $[0, 1]$, leaving two line segments: $[0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$. Next, the open middle third of each of these remaining segments is deleted, leaving four line segments: $[0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{7}{9}] \cup [$

$\frac{8}{9}$, 1]. This process is continued ad infinitum, where the nth set is

$$C_n = \frac{C_{n-1}}{3} \cup \left(\frac{2}{3} + \frac{C_{n-1}}{3}\right) \text{ for } n \geq 1, \text{ and } C_0 = [0,1].$$

The Cantor ternary set contains all points in the interval [0, 1] that are not deleted at any step in this infinite process:

$$\mathcal{C} := \bigcap_{n=1}^{\infty} C_n.$$

The first six steps of this process are illustrated below.

Using the idea of self-similar transformations, $T_L(x) = x/3, T_R(x) = (2+x)/3$ and $C_n = T_L(C_{n-1}) \cup T_R(C_{n-1})$, the explicit closed formulas for the Cantor set are[7]

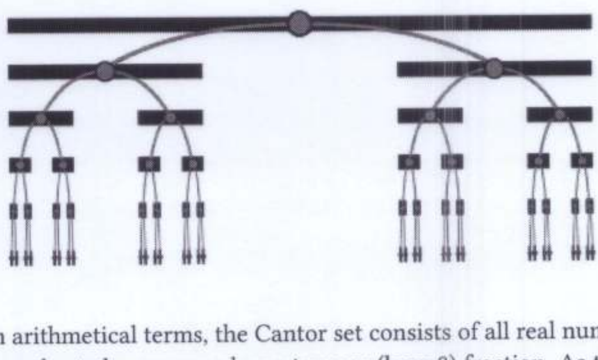
$$\mathcal{C} = [0,1] \setminus \bigcup_{n=0}^{\infty} \bigcup_{k=0}^{3^n-1} \left(\frac{3k+1}{3^{n+1}}, \frac{3k+2}{3^{n+1}}\right),$$

where every middle third is removed as the open interval $\left(\frac{3k+1}{3^{n+1}}, \frac{3k+2}{3^{n+1}}\right)$ from the closed interval $\left[\frac{3k+0}{3^{n+1}}, \frac{3k+3}{3^{n+1}}\right] = \left[\frac{k+0}{3^n}, \frac{k+1}{3^n}\right]$ surrounding it, or

$$\mathcal{C} = \bigcap_{n=1}^{\infty} \bigcup_{k=0}^{3^{n-1}-1} \left(\left[\frac{3k+0}{3^n}, \frac{3k+1}{3^n}\right] \cup \left[\frac{3k+2}{3^n}, \frac{3k+3}{3^n}\right]\right),$$

where the middle third $\left(\frac{3k+1}{3^n}, \frac{3k+2}{3^n}\right)$ of the foregoing closed interval $\left[\frac{k+0}{3^{n-1}}, \frac{k+1}{3^{n-1}}\right] = \left[\frac{3k+0}{3^n}, \frac{3k+3}{3^n}\right]$ is removed by intersecting with $\left[\frac{3k+0}{3^n}, \frac{3k+1}{3^n}\right] \cup \left[\frac{3k+2}{3^n}, \frac{3k+3}{3^n}\right]$.

This process of removing middle thirds is a simple example of a finite subdivision rule. The Cantor ternary set is an example of a fractal string.



In arithmetical terms, the Cantor set consists of all real numbers of the unit interval $[0,1]$ that do not require the digit 1 in order to be expressed as a ternary (base 3) fraction. As the above diagram illustrates, each point in the Cantor set is uniquely located by a path through an infinitely deep binary tree, where the path turns left or right at each level according to which side of a deleted segment the point lies on. Representing each left turn with 0 and each right turn with 2 yields the ternary fraction for a point. Replacing the "2" digits in these fractions with "1" digits produces a surjective (and not injective) mapping between the Cantor set and the set of binary fractions in the interval [0,1].

Composition

Since the Cantor set is defined as the set of points not excluded, the proportion (i.e., measure) of the unit interval remaining can be found by total length removed. This total is the geometric progression

$$\sum_{n=0}^{\infty} \frac{2^n}{3^{n+1}} = \frac{1}{3} + \frac{2}{9} + \frac{4}{27} + \frac{8}{81} + \cdots = \frac{1}{3}\left(\frac{1}{1-\frac{2}{3}}\right) = 1.$$

So that the proportion left is 1 – 1 = 0.

This calculation suggests that the Cantor set cannot contain any interval of non-zero length. It may seem surprising that there should be anything left—after all, the sum of the lengths of the removed intervals is equal to the length of the original interval. However, a closer look at the process reveals that there must be something left, since removing the "middle third" of each interval involved removing open sets (sets that do not include their endpoints). So removing the line segment (⅓, ⅔) from the original interval [0, 1] leaves behind the points ⅓ and ⅔. Subsequent steps do not remove these (or other) endpoints, since the intervals removed are always internal to the intervals remaining. So the Cantor set is not empty, and in fact contains an uncountably infinite number of points (as follows from the above description in terms of paths in an infinite binary tree).

It may appear that *only* the endpoints of the construction segments are left, but that is not the case either. The number ¼, for example, has the unique ternary form $0.020202... = 0.\overline{02}$. It is in the bottom third, and the top third of that third, and the bottom third of that top third, and so on. Since it is never in one of the middle segments, it is never removed. Yet it is also not an endpoint of any middle segment, because it is not a multiple of any power of 1/3.[8] All endpoints of segments are *terminating* ternary fractions and are contained in the set

$$\{x \in [0,1] \mid \exists i \in \mathbb{N}_0 : x\,3^i \in \mathbb{Z}\} \qquad \left(\subset \mathbb{N}_0\,3^{-\mathbb{N}_0}\right)$$

which is a countably infinite set. As to cardinality, almost all elements of the Cantor set are not endpoints of intervals, and the whole Cantor set is not countable.

Properties

Cardinality

It can be shown that there are as many points left behind in this process as there were to begin with, and that therefore, the Cantor set is uncountable. To see this, we show that there is a function *f* from the Cantor set $\mathcal{C}$ to the closed interval [0,1] that is surjective (i.e. *f* maps from $\mathcal{C}$ onto [0,1]) so that the cardinality of $\mathcal{C}$ is no less than that of [0,1]. Since $\mathcal{C}$ is a subset of [0,1], its cardinality is also no greater, so the two cardinalities must in fact be equal, by the Cantor–Bernstein–Schröder theorem.

To construct this function, consider the points in the [0, 1] interval in terms of base 3 (or ternary) notation. Recall that the proper ternary fractions, more precisely: the elements of $(\mathbb{Z} \setminus \{0\}) \cdot 3^{-\mathbb{N}_0}$, admit more than one representation in this notation, as for example ⅓, that can be written as $0.1_3 = 0.1\overline{0}_3$, but also as $0.0222..._3 = 0.0\overline{2}_3$, and ⅔, that can be written as $0.2_3 = 0.2\overline{0}_3$ but also as $0.1222..._3 = 0.1\overline{2}_3$.[9] When we remove the middle third, this contains the numbers with ternary numerals of the form $0.1xxxxx..._3$ where $xxxxx..._3$ is strictly between $00000..._3$ and $22222..._3$. So the numbers remaining after the first step consist of

- Numbers of the form $0.0xxxxx..._3$ (including $0.022222..._3 = 1/3$)
- Numbers of the form $0.2xxxxx..._3$ (including $0.222222..._3 = 1$)

This can be summarized by saying that those numbers with a ternary representation such that the first digit after the

Δυναμοσυνολο του E

Θα δείξουμε οτι το δυναμοσυνολο του [n] περιεχει 2^n στοιχεια, $n \geq 1$

Με επαγωγη

Για $n=1$ $\quad [1]=\{1\}$

$$P([1]) = \{\emptyset, \{1\}\}$$

$$|P([1])| = 2^1 = 2$$

Εστω ότι ~~για n=k~~ η πρόταση ισχυει για το $n-1$, δηλαδη

$$|P([n-1])| = 2^{n-1}$$

Θα δειξουμε οτι η προταση ισχυει για το n, δηλαδη

$$|P([n])| = 2^n$$

Παρατηρούμε ότι τα $X \in P([n])$ διαμερίζονται σε δυο κατηγορίες

- Τα X που περιέχουν το n
- Τα X που δεν περιέχουν το n

Τα X που δεν περιέχουν το n ανήκουν στο $P([n-1])$ διότι $X \subseteq [n-1]$ και αντίστροφα.

Τα X που περιέχουν το n αν αφαιρέσω το n θα προκύψει ένα σύνολο X' που ανήκει στο $P([n-1])$ διότι $X' \subseteq [n-1]$ και αντίστροφα.

Άρα, το $P([n])$ διαμερίζεται σε ένα σύνολο A που είναι ισοδύναμο με το $P([n-1])$ και σε ένα άλλο σύνολο B που πάλι είναι ισοδύναμο $P([n-1])$

$$
\begin{aligned}
|P(n)| &= |A \cup B| \\
&= |A| + |B| \\
&= 2^{n-1} + 2^{n-1} \\
&= 2 \cdot 2^{n-1} = 2^{n}
\end{aligned}
$$

		3			6			9	
a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}
1	1	2	3	5	8	13	21	34	55

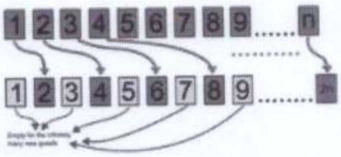
$$\sum_{1\le i<j\le 3} \alpha_{ij} = \sum_{i=1}^{2}\sum_{j=i+1}^{3}$$

$$= \underbrace{\alpha_{12} + \alpha_{13}}_{i=1} + \underbrace{\alpha_{23}}_{i=2}$$

$$\sum_{1\le i<j\le 4} \alpha_{ij} = \sum_{i=1}^{3}\sum_{j=i+1}^{4} \alpha_{ij}$$

$$= \underbrace{\alpha_{12} + \alpha_{13} + \alpha_{14}}_{i=1} + \underbrace{\alpha_{23} + \alpha_{24}}_{i=2} + \underbrace{\alpha_{34}}_{i=3}$$

Hilbert's paradox of the Grand Hotel



Hilbert's paradox of the Grand Hotel (colloquial: **Infinite Hotel Paradox** or **Hilbert's Hotel**) is a thought experiment which illustrates a counterintuitive property of infinite sets. It is demonstrated that a fully occupied hotel with infinitely many rooms may still accommodate additional guests, even infinitely many of them, and this process may be repeated infinitely often. The idea was introduced by David Hilbert in a 1924 lecture "Über das Unendliche", reprinted in (Hilbert 2013, p.730), and was popularized through George Gamow's 1947 book *One Two Three... Infinity*.[1][2]

The paradox[edit]

Consider a hypothetical hotel with a countably infinite number of rooms, all of which are occupied. One might be tempted to think that the hotel would not be able to accommodate any newly arriving guests, as would be the case with a finite number of rooms, where the pigeonhole principle would apply.

Finitely many new guests[edit]

Suppose a new guest arrives and wishes to be accommodated in the hotel. We can (simultaneously) move the guest currently in room 1 to room 2, the guest currently in room 2 to room 3, and so on, moving every guest from their current room n to room n+1. After this, room 1 is empty and the new guest can be moved into that room. By repeating this procedure, it is possible to make room for any finite number of new guests.

Infinitely many new guests[edit]

It is also possible to accommodate a *countably infinite* number of new guests: just move the person occupying room 1 to room 2, the guest occupying room 2 to room 4, and, in general, the guest occupying room n to room $2n$ (2 times n), and all the odd-numbered rooms (which are countably infinite) will be free for the new guests.

Infinitely many coaches with infinitely many guests each[edit]

It is possible to accommodate countably infinitely many coachloads of countably infinite passengers each, by several different methods. Most methods depend on the seats in the coaches being already numbered (or use the axiom of countable choice). In general any pairing function can be used to solve this problem. For each of these methods, consider a passenger's seat number on a coach to be n, and their coach number to be c, and the numbers n and c are then fed into the two arguments of the pairing function.

Prime powers method[edit]

Empty the odd numbered rooms by sending the guest in room i to room 2^i, then put the first coach's load in rooms 3^n, the second coach's load in rooms 5^n; for coach number c we use the rooms p^n where p is the cth odd prime number. This solution leaves certain rooms empty (which may or may not be useful to the hotel); specifically, all odd numbers that are not prime powers, such as 15 or 847,