Exponential Distribution

- The PDF for the exponential distribution is parameterized by its rate parameter $\lambda > 0$ and is given by:

$$\boxed{f(x;\lambda) = \lambda \cdot e^{-\lambda x}, \quad x \geq 0 \quad [1]}$$

- The expected value of a random variable X that follows the exponential distribution will be given as:

$$E[X] = \int_0^{+\infty} x \cdot f(x;\lambda)\,dx = \int_0^{+\infty} x \cdot \lambda \cdot e^{-\lambda x}\,dx \quad [2]$$

- The integral appearing in Eq. (2) may be handled by utilizing the technique of integration by parts:

$$\int_0^{+\infty} x \cdot \lambda \cdot e^{-\lambda x} \cdot dx = -\int_0^{+\infty} -x \cdot \lambda \cdot e^{-\lambda x}\,dx = -\int_0^{+\infty} x \cdot (-\lambda \cdot e^{-\lambda x})\,dx \Rightarrow$$

$$E[X] = \int_0^{+\infty} -x \cdot (e^{-\lambda x})'\,dx = \left[-x \cdot e^{-\lambda x}\right]_0^{+\infty} - \int_0^{+\infty} e^{-\lambda x} \cdot (-x)'\,dx \Rightarrow$$

$$\boxed{E[X] = \left[-x e^{-\lambda x}\right]_0^{+\infty} - \int_0^{+\infty} -e^{-\lambda x}\,dx} \iff$$

$$\boxed{E[X] = \left[-x e^{-\lambda x}\right]_0^{+\infty} + \int_0^{+\infty} e^{-\lambda x}\,dx \quad [3]}$$

(#2)

(*) It is easy to see that: $\left[-xe^{-\lambda x}\right]_0^{+\infty} = \left[xe^{-\lambda x}\right]_{+\infty}^{0} \Rightarrow$

$$\boxed{0 \cdot e^{0} - \lim_{x \to +\infty} x \cdot e^{-\lambda x} = \left[xe^{-\lambda x}\right]_{+\infty}^{0} \quad [4]}$$

(*) The required limit is of the form $+\infty * 0$ which can be handled as:

$$\lim_{x \to +\infty} x \cdot e^{-\lambda x} \overset{+\infty \cdot 0}{=\!=} \lim_{x \to +\infty} \frac{x}{e^{\lambda x}} = \frac{+\infty}{+\infty} =$$

$$\overset{\text{De L'Hopital}}{=\!=} \lim_{x \to +\infty} \frac{x'}{(e^{\lambda x})'} = \lim_{x \to +\infty} \frac{1}{\lambda \cdot e^{\lambda x}} = \frac{1}{\infty} = 0 \quad [5]$$

(*) Therefore, according to Eq. (2), we have that:

$$E[X] = \int_0^{+\infty} e^{-\lambda x} dx = \int_0^{+\infty} \left(-\frac{1}{\lambda} \cdot e^{-\lambda x}\right)' dx \Rightarrow$$

$$E[X] = \left[-\frac{1}{\lambda} \cdot e^{-\lambda x}\right]_0^{+\infty} = \left[\frac{1}{\lambda} e^{-\lambda x}\right]_{+\infty}^{0} \Rightarrow$$

$$E[X] = \frac{1}{\lambda} \cdot e^{0} - \frac{1}{\lambda} \cdot e^{\overset{\to 0}{-\infty}} \Rightarrow \boxed{E[X] = \frac{1}{\lambda} \quad [6]}$$

⊛ Finally, the variance of a random variable that follows an exponential distribution will be given as:

$$\boxed{Var[X] = E[X^2] - E[X]^2 \quad [7]}$$

⊛ Thus, we need to compute the second order moment for the random variable X as:

$$E[X^2] = \int_0^{\infty} x^2 \cdot f(x;\lambda)\,dx = \int_0^{+\infty} x^2 \cdot \lambda \cdot e^{-\lambda x}\,dx \quad [8]$$

⊛ $E[X^2] = \int_0^{+\infty} -x^2 \cdot (-\lambda \cdot e^{-\lambda x})\,dx = \int_0^{+\infty} -x^2 \cdot (e^{-\lambda x})'\,dx \Rightarrow$

$$E[X^2] = [-x^2 \cdot e^{-\lambda x}]_0^{+\infty} - \int_0^{+\infty} e^{-\lambda x} \cdot (-x^2)'\,dx \Rightarrow$$

$$\boxed{E[X^2] = [-x^2 \cdot e^{-\lambda x}]_0^{+\infty} + \int_0^{+\infty} 2 \cdot x \cdot e^{-\lambda x}\,dx \quad [9]}$$

⊛ $[-x^2 \cdot e^{-\lambda x}]_0^{+\infty} = [x^2 e^{-\lambda x}]_{+\infty}^{0} = 0 \cdot e^0 - \lim_{x \to +\infty} x^2 \cdot e^{-\lambda x} \quad [10]$

⊛ The required limit can be obtained as:

$$\lim_{x \to +\infty} x^2 \cdot e^{-\lambda x} \overset{+\infty \cdot 0}{=\!=} \lim_{x \to +\infty} \frac{x^2}{e^{\lambda x}} \overset{+\infty/+\infty}{=\!=} \lim_{x \to +\infty} \frac{2x}{\lambda \cdot e^{\lambda x}} = \frac{2}{\lambda} \cdot \lim_{x \to +\infty} \frac{x}{e^{\lambda x}} = \frac{+\infty}{+\infty}$$

$$= \frac{2}{\lambda} \cdot \lim_{x \to +\infty} \frac{x'}{(e^{\lambda x})'} = \frac{2}{\lambda} \cdot \lim_{x \to +\infty} \frac{1}{\lambda \cdot e^{\lambda x}} = \frac{2}{\lambda^2} \cdot \frac{1}{+\infty} = 0 \quad [11]$$

- Eq. (9) yields : $E[x^2] = 2\int_0^{+\infty} x \cdot e^{-\lambda x} dx \Rightarrow$

$$E[x^2] = \frac{2}{\lambda}\int_0^{+\infty} x \cdot \lambda \cdot e^{-\lambda x} dx \Rightarrow \boxed{E[x^2] = \frac{2}{\lambda} E[X] \quad [12]}$$

- Eq. (6), thus provides: $\boxed{E[x^2] = \frac{2}{\lambda^2} \quad [13]}$

- Finally, Eq. (7) $\xrightarrow[Eq.(13)]{Eq.(6)}$ $Var[X] = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2} \Rightarrow$

$$\boxed{Var[X] = \frac{1}{\lambda^2} \quad [14]}$$

Memoryless Property of the Exponential Distribution

(*) The memoryless property of the exponential distribution is an important characteristic that sets it apart from other continuous probability distributions. It states that:

$$\boxed{P(x > s+t \mid x > s) = P(x > t) \quad [15]}$$

$\forall\, s, t \geq 0.$

- This means that the probability that the process will continue for an additional time t does not depend on how much time s has elapsed.

Derivation of the Memoryless Property of the Exponential Distribution:

(i): We need to express the quantity $P(X > s+t \mid X > s)$ utilizing the definition of the conditional probability

$$\boxed{P(X > s+t \mid X > s) \overset{P(X|Y) = \frac{P(X,Y)}{P(Y)}}{=\!=\!=} \frac{P(X > s+t,\, X > S)}{P(X > S)} \quad [II]}$$

(ii) It is easy to deduce that: $\boxed{X > s+t \Rightarrow X > s}$

Thus, satisfying the first condition suffices for satisfying the second condition. Therefore, we may write that:

$$\boxed{P(X > s+t,\, X > s) = P(X > s+t) \quad [III]}$$

(iii) In light of the previous derivations, we can re-write Eq. (II) as:

$$\boxed{P(X > s+t \mid X > s) = \frac{P(X > S+t)}{P(X > S)} \quad [IV]}$$

(iv): For an exponentially distributed random variable, we know that $P(X > x) = 1 - F_X(x) \Rightarrow$

$$\boxed{P(X > x) = e^{-\lambda x} \quad (V)}$$

(v): Taking into consideration Eq. (v), Eq. (IV) yields:

$$\boxed{P(X > s+t \mid X > s) = \frac{e^{-\lambda(s+t)}}{e^{-\lambda s}} = e^{-\lambda t} \quad (VI)}$$

(vi): Finally, having in mind Eq. (V), Eq. (VI) gives that:

$$\boxed{P(X > s+t \mid X > s) = e^{-\lambda t} = P(X > t) \quad (VII)}$$

Exponential Distribution

$$\boxed{f_X(x) = \lambda \cdot e^{-\lambda x}, \quad x \geq 0 \text{ and } \lambda > 0 \qquad (1)}$$

$$\boxed{F_X(x) = P(X \leq x) = 1 - e^{-\lambda x}, \quad x \geq 0 \qquad (2)}$$

$$\boxed{P(X > x) = e^{-\lambda x} \qquad (3)}$$

$$\boxed{\mu = E[X] = \frac{1}{\lambda} \quad (4)} \qquad \boxed{\sigma^2 = Var[X] = \frac{1}{\lambda^2} \quad (5)}$$

Memoryless Property

$$\boxed{P(X > t+s \mid X > t) = P(X > s) \quad (6)} \qquad s, t \geq 0$$

(*) Η πιθανότητα η Τ.Μ. X να υπερβεί την τιμή $t+s$, με $0 < t < t+s$, δοθέντος ότι έχει υπερβεί την τιμή t, είναι ανεξάρτητη του t και ίση με την αδέσμευτη πιθανότητα να υπερβεί την τιμή s.

Answer

(i): Express the given conditional probability:

$$\boxed{P(X > t+s \mid X > t) \xrightarrow[\text{BAYES RULE}]{P(X|Y) = \frac{P(X,Y)}{P(Y)}} \frac{P(X > t+s,\ X > t)}{P(X > t)} \quad (7)}$$

⊛ We may identify that $\boxed{X > t+s \Rightarrow X > t}$

⊛ If X is greater than the sum t+s, it will also be greater than t alone. Otherwise, we may say that if the first condition is satisfied, the second condition will also be satisfied.

⊛ Thus, we may write that:

$$\boxed{P(X > t+s,\ X > t) = P(X > t+s) \quad (8)}$$

⊛ According to Eq. (7) and (8), we may write that:

$$\boxed{P(X > t+s \mid X > t) = \frac{P(X > t+s)}{P(X > t)} = \frac{e^{-\lambda(t+s)}}{e^{-\lambda t}} = e^{-\lambda s} \quad (9)}$$

⊛ But, this is $\boxed{P(X > s) = e^{-\lambda s} \quad (10).}$

⊛ We may finally conclude that:

$$\boxed{P(X > t+s \mid X > t) = P(X > s) \quad (11)}$$

Problem: A service center has two independent (#3) service desks where customers are attended to, and the service times at both desks follow an exponential distribution.

The time a customer spends being served at desk 1 follows an exponential distribution with $\mu_1 = \frac{1}{\lambda_1}$, and the time a customer spends being served at desk 2 follows an exponential distribution with $\mu_2 = \frac{1}{\lambda_2}$.

A customer initially enters desk 1 and starts being served. If the service takes longer than a certain threshold time T, the customer abandons desk 1 and switches to desk 2, where they continue their service without returning to desk 1.

If the customer switches from desk 1 to desk 2, the service time at desk 2 starts anew and is independent of the time already spent at desk 1.

Questions: (I): Calculate the probability that a customer will leave desk 1 before their service is completed.

(II): Determine the expected time (total time) a customer spends in the system, whether they finish at desk 1 or desk 2 and complete their service there.

(III): Find the variance of the total time.

Solution: X_1: time needed to complete the service at D_1 (#4)

X_2: time needed to complete the service at D_2

Problem Recap:

$$\begin{cases} \text{Service at desk 1: } X_1 \sim Exp(\lambda_1) \\ \text{Service at desk 2: } X_2 \sim Exp(\lambda_2) \\ \text{Threshold Time: } T\text{, determines whether a customer switches to desk 2.} \end{cases}$$

Q_1: Probability of switching from D_1 to D_2:

We need to calculate the probability of the waiting time at desk 1 to exceed the threshold value T:

$$\boxed{P(X_1 > T) = e^{-\lambda_1 T} \quad (1)}$$

Q_2: Expected Total Time in the System $E[W]$:

Total Service Time:

(A): If the customer completes the service at desk 1, the total service time is simply $\boxed{W = X_1 \text{ if } X_1 \le T.}$

(B): If the customer switches to desk 2, because $X_1 > T$, the total service time becomes $\boxed{W = T + X_2\text{, when } X_1 > T.}$

Case A: This case happens with probability:

$$\boxed{P(X_1 \leq T) = 1 - e^{-\lambda_1 T} \quad (2)}$$

- Thus, the corresponding expected time for the termination the service at desk 1, can be represented as: $\{W = X_1\}$

$$\boxed{E[X_1 \mid X_1 \leq T] = \int_0^T x \cdot f_{X_1 \mid X_1 \leq T}(x)\, dx \quad (3)}$$

where $f_{X_1 \mid X_1 \leq T}(x)$ is the conditional PDF given by:

$$\boxed{f_{X_1 \mid X_1 \leq T}(x) = \frac{f_{X_1}(x)}{P(X_1 \leq T)} = \frac{1}{1 - e^{-\lambda_1 T}} \cdot \lambda_1 \cdot e^{-\lambda_1 x} \quad (4)}$$

- Combining, Eqs. (3) and (4) yields:

$$\boxed{E[X_1 \mid X_1 \leq T] = \frac{1}{1 - e^{-\lambda_1 T}} \cdot \int_0^T x \cdot \lambda_1 \cdot e^{-\lambda_1 x}\, dx \quad (4)}$$

- The integral $I = \int_0^T x \cdot \lambda_1 \cdot e^{-\lambda_1 x} dx$ can be computed threw integration by parts as:

$$I = \int_0^T -x \cdot (-\lambda_1 \cdot e^{-\lambda_1 x})dx = \int_0^T -x \cdot (e^{-\lambda_1 x})' dx = \left[-x \cdot e^{-\lambda_1 x}\right]_0^T - \int_0^T (-x)' e^{-\lambda_1 x} dx \Rightarrow$$

$$I = \left[-x e^{-\lambda_1 x}\right]_0^T + \int_0^T e^{-\lambda_1 x} dx = \left[-x \cdot e^{-\lambda_1 x}\right]_0^T + \int_0^T \left(-\frac{1}{\lambda_1} e^{-\lambda_1 x}\right)' dx \Rightarrow$$

$$I = \left[-x e^{-\lambda_1 x}\right]_0^T + \left[-\frac{1}{\lambda_1} e^{-\lambda_1 x}\right]_0^T \Rightarrow$$

$$\boxed{I = \left[x \cdot e^{-\lambda_1 x}\right]_T^0 + \left[\frac{1}{\lambda_1} e^{-\lambda_1 x}\right]_T^0 \quad (5)}$$

⊛ The final result for the integral I, will be given (#6) as:

$$\boxed{I = -T \cdot e^{-\lambda_1 T} + \frac{1 - e^{-\lambda_1 T}}{\lambda_1} \quad (6A)} \Rightarrow$$

$$I = \frac{1}{\lambda_1} - T \cdot e^{-\lambda_1 T} - \frac{1}{\lambda_1} \cdot e^{-\lambda_1 T} \rightarrow$$

$$\boxed{I = \frac{1}{\lambda_1} - \left(\frac{1}{\lambda_1} + T\right) \cdot e^{-\lambda_1 T} \quad (6_B)}$$

⊛ Therefore, the conditional expected value will be given as:

$$(4) \xrightarrow[(6A)]{} \dots$$

$$E[X_1 | X_1 \leq T] = \frac{I}{1 - e^{-\lambda_1 T}} \Rightarrow$$

$$E[X_1 | X_1 \leq T] = -\frac{T \cdot e^{-\lambda_1 T}}{1 - e^{-\lambda_1 T}} + \frac{1}{\lambda_1} \quad \text{or}$$

$$\boxed{E[X_1 | X_1 \leq T] = \frac{1}{\lambda_1} - \frac{T \cdot e^{-\lambda_1 T}}{1 - e^{-\lambda_1 T}} \quad (7)}$$

Case B: This case happens with probability $P(X_1 > T)$:

$$\boxed{P(X_1 > T) = 1 - P(X_1 \leq T) = e^{-\lambda_1 T} \quad (8)}$$

⊛ The corresponding expected time for the termination of the service after switching to desk 2 can be represented as:

$\{W = T + X_2\}$ (given that $X_1 > T$):

$$\boxed{E[W | X_1 > T] = T + E[X_2] \quad (9)}$$

⊛ Given that $X_2 \sim Exp(\lambda_2) \Rightarrow$ $\boxed{E[X_2] = \frac{1}{\lambda_2}}$ (10)

⊛ Combining Eqs. (9) and (10) yields:

$$\boxed{E[W \mid X_1 > T] = T + \frac{1}{\lambda_2} \quad (11)}$$

TOTAL EXPECTED SERVICE TIME:

$$E[W] = P(X_1 \leq T) \cdot E[\overset{W}{\overset{\shortparallel}{X_1}} \mid X_1 \leq T] + P(X_1 > T) \cdot E[W \mid X_1 > T]$$

$$\Longrightarrow$$

$$\boxed{E[W] = (1 - e^{-\lambda_1 T}) \cdot \left(\frac{1}{\lambda_1} - \frac{T \cdot e^{-\lambda_1 T}}{1 - e^{-\lambda_1 T}} \right) + e^{-\lambda_1 T} \cdot \left(T + \frac{1}{\lambda_2} \right) \quad (12)}$$