Συνάρτηση Γάμμα

Η συνάρτηση $F: \mathbb{R}^+ \to \mathbb{R}^+$ ορίζεται ως εξής:

$$\Gamma(x) = \int_0^{+\infty} t^{x-1} \cdot e^{-t} \, dt \qquad \boxed{x>0} \qquad (\mathrm{I})$$

ΙΔΙΟΤΗΤΕΣ ΣΥΝΑΡΤΗΣΗΣ ΓΑΜΜΑ:

(i): $\Gamma(x) = \Gamma(x-1)\cdot(x-1)$

(ii): $\Gamma(n) = (n-1)!$

(iii): $\Gamma(1/2) = \sqrt{\pi}$

(iv): $\Gamma(n+\frac{1}{2}) = \sqrt{\pi}\cdot\prod_{j=0}^{n-1}(j+\frac{1}{2})$

(II)

Gamma Distribution:

The Gamma Distribution can be parameterized in terms of a scale parameter (a) and an inverse scale parameter (β), also called rate parameter

$X \sim \Gamma(a,\beta) \Leftrightarrow$

$$f_X(x) = \frac{x^{a-1}\cdot e^{-\beta x}\cdot \beta^{a}}{\Gamma(a)} \qquad x \geq 0,\ a>0,\ \beta>0 \qquad (\mathrm{III})$$

- **Expected Value:**

$$E[X] = \int_0^{+\infty} x \cdot f_X(x)\,dx = \int_0^{+\infty} x \cdot \frac{x^{a-1} \cdot e^{-\beta x} \cdot \beta^a}{\Gamma(a)} \cdot dx \rightarrow$$

$$\boxed{E[X] = \frac{\beta^a}{\Gamma(a)} \int_0^{+\infty} x^a \cdot e^{-\beta x}\,dx \quad (1)}$$

Consider the following transformation:

$$u = \beta x \Rightarrow x = \frac{1}{\beta} u \Rightarrow \boxed{dx = \frac{1}{\beta} du \quad (2)}$$

$$E[X] = \frac{\beta^a}{\Gamma(a)} \int_0^{+\infty} \left(\frac{u}{\beta}\right)^a \cdot e^{-u} \cdot \frac{du}{\beta} \rightarrow$$

$$E[X] = \frac{\beta^a}{\beta^{a+1}\Gamma(a)} \int_0^{+\infty} u^a \cdot e^{-u} \cdot du \quad (3) \rightarrow$$

$$\boxed{E[X] = \frac{1}{\beta\Gamma(a)} \cdot \int_0^{+\infty} u^a \cdot e^{-u} \cdot du = \frac{\Gamma(a+1)}{\beta \cdot \Gamma(a)} \quad (4)}$$

(*) We know that: $\Gamma(a+1) = a\Gamma(a)$

$$E[X] = \frac{a\Gamma(a)}{\beta\Gamma(a)} \Rightarrow \boxed{E[X] = \frac{a}{\beta} \quad (5)}$$

- **Varionie:** Estimateing the varianie requires computing $E[X^2]$.

$$E[X^2] = \int_0^{+\infty} x^2 \cdot f_X(x)\,dx = \int_0^{+\infty} x^2 \cdot \frac{x^{a-1} \cdot e^{-\beta x} \cdot \beta^{a}}{\Gamma(a)}\,dx \Rightarrow$$

$$E[X^2] = \frac{\beta^a}{\Gamma(a)} \cdot \int_0^{+\infty} x^{a+1} \cdot e^{-\beta x}\,dx \quad (6)$$

Consider the same transformation:

$$\boxed{u = \beta x \Rightarrow x = \frac{1}{\beta} u \rightarrow dx = \frac{1}{\beta} du}$$

$$E[X^2] = \frac{\beta^a}{\Gamma(a)} \cdot \int_0^{+\infty} \left(\frac{u}{\beta}\right)^{a+1} \cdot e^{-u} \cdot \frac{du}{\beta} \rightarrow$$

$$E[X^2] = \frac{\beta^a}{\beta^{a+2}\,\Gamma(a)} \int_0^{+\infty} u^{a+1} \cdot e^{-u}\,du \rightarrow$$

$$E[X^2] = \frac{\Gamma(a+2)}{\beta^2 \cdot \Gamma(a)} \xrightarrow[= (a+1) \cdot a \cdot \Gamma(a)]{\Gamma(a+2) = (a+1)\Gamma(a+1) =}$$

$$E[X^2] = \frac{a(a+1)\,\Gamma(a)}{\beta^2\,\Gamma(a)} \rightarrow$$

$$\boxed{E[X^2] = \frac{a(a+1)}{\beta^2} \quad (7)}$$

Finally, the variance will be given as:

$$Var[X] = E[X^2] - E^2[X] \Rightarrow$$

$$Var[X] = \frac{a(a+1)}{\beta^2} - \left(\frac{a}{\beta}\right)^2 \Rightarrow$$

$$Var[X] = \frac{a^2 + a - a^2}{\beta^2} \Rightarrow$$

$$\boxed{Var[X] = \frac{a}{\beta^2} \quad (8)}$$

Moment Generating Function:

- The moment generating function $M_X(t)$ for a random variable X is defined as:

$$\boxed{M_X(t) = E[e^{tX}] = \int_{-\infty}^{+\infty} e^{tx} \cdot f_X(x)\,dx \quad (A)}$$

where $f_X(x)$ is the probability density function of X, given that the expectation exists for t in some neighborhood of 0.

(★) The moment generating function (MGF) (Ροπογεννήτρια Συνάρτηση) is so called because it can be used to find the moments of the underlying distribution. Considering the Taylor series expansion of e^{tX}:

$$\boxed{e^{tX} = 1 + t \cdot X + \frac{t^2 X^2}{2!} + \frac{t^3 X^3}{3!} + \cdots + \frac{t^n X^n}{n!} + \cdots \quad (B)}$$

Thus,

$$\boxed{E[e^{tX}] = 1 + t\,E[X] + \frac{t^2}{2!} E[X^2] + \frac{t^3}{3!} E[X^3] + \cdots + \frac{t^n}{n!} E[X^n] + \cdots \quad (\Gamma)}$$

- The n-th term $m_n = E[X^n]$ identifies the n-th moment of X.

- The last formula suggests that:

$$\boxed{M_X(t) = \sum_{n=0}^{\infty} \frac{t^n}{n!} m_n \quad (A) \quad \text{where} \quad m_n = E[X^n] \quad (E)}$$

- The previous formulation suggests that if we are in position of acquiring the power series expansion of $M_X(t)$ in the following way:

$$\boxed{M_X(t) = \sum_{n=0}^{\infty} a_n t^n \quad (Z)}$$

then by equating the respective terms of the two power series, we get that:

$$m_n \cdot \frac{t^n}{n!} = a_n t^n \Rightarrow \boxed{m_n = n! \, a_n \ (H)}$$

★ THEOREM: If random variable X has mgf $M_X(t)$, then $\boxed{E[X^n] = M_X^{(n)}(0)}$ where

$$\boxed{M_X^{(n)}(0) = \frac{d^n}{dt^n} M_X(t) \Big|_{t=0} \quad (\theta)}$$

ΑΣΚΗΣΗ: Να βρεθεί η Ροπογεννήτρια συνάρτηση μιας Τ.Μ. που ακολουθεί την κατανομή Γάμμα.

$$M_X(t) = ? \quad \text{όταν} \quad X \sim \text{Γαμμα}(\alpha, \beta).$$

ΛΥΣΗ: Γνωρίζουμε ότι: $f_X(x;\alpha,\beta) = \dfrac{x^{\alpha-1} \cdot e^{-\beta x} \cdot \beta^{\alpha}}{\Gamma(\alpha)}, \quad x > 0.$

Σύμφωνα με τον ορισμό της ροπογεννήτριας συνάρτησης θα έχουμε ότι:

$$\boxed{M_X(t) = E[e^{tX}] = \int_0^{+\infty} e^{tx} \cdot x^{\alpha-1} \cdot e^{-\beta x} \cdot \frac{\beta^{\alpha}}{\Gamma(\alpha)}\, dx \quad (1)}$$

Συνδυάζοντας τους εκθετικούς όρους έχουμε ότι:

$$\boxed{M_X(t) = \frac{\beta^{\alpha}}{\Gamma(\alpha)} \cdot \int_0^{+\infty} x^{\alpha-1} \cdot e^{-(\beta-t)x} \cdot dx \quad (2)}$$

S.O.S The integral in Eq.(2) converges only when $\beta - t > 0$ or $t < \beta$.

Under these conditions, we may use the following transformation:

$$(\beta - t)\cdot x = u \Rightarrow x = \frac{u}{\beta - t} \Rightarrow \boxed{dx = \frac{du}{\beta - t} \quad (3)}$$

Eq.(2), then, provides:

$$M_X(t) = \frac{\beta^{\alpha}}{\Gamma(\alpha)} \int_0^{+\infty} \left(\frac{u}{\beta - t}\right)^{\alpha-1} \cdot e^{-u} \cdot \frac{du}{\beta - t} \Rightarrow$$

$$\boxed{M_X(t) = \frac{\beta^{\alpha}}{(\beta - t)^{\alpha}\,\Gamma(\alpha)} \int_0^{+\infty} u^{\alpha-1} \cdot e^{-u}\, du = \frac{\beta^{\alpha}\,\Gamma(\alpha)}{(\beta - t)^{\alpha}\,\Gamma(\alpha)} = \left(\frac{\beta}{\beta - t}\right)^{\alpha} \quad (3)}$$