Beta Function

$$\boxed{B(x,y) = \int_0^1 t^{x-1} \cdot (1-t)^{y-1}\, dt \quad , \quad x>0,\ y>0 \quad (I)}$$

Relation to the Gamma Function:

$$\boxed{B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \quad (II)}$$

Beta Distribution:

Beta Distribution is a continuous probability distribution defined on the $[0,1]$ interval, characterized by two shape parameters $\alpha>0$ and $\beta>0$.

The PDF of the Beta Distribution is given by:

$$\boxed{f_X(x;\alpha,\beta) = \frac{x^{\alpha-1}(1-x)^{\beta-1}}{B(\alpha,\beta)} \quad , \quad 0 \leq x \leq 1 \quad (III)}$$

★ Derive the expectation $E[X]$ and variance $Var[X]$ of a random variable $X \sim Beta(\alpha,\beta)$

(a): By definition of the expectation, we have that:

$$E[X] = \int_0^1 x \cdot f_X(x; a, \beta)\, dx = \int_0^1 x \cdot \frac{x^{a-1} (1-x)^{\beta-1}}{B(a,\beta)} \cdot dx \Rightarrow$$

$$E[X] = \frac{1}{B(a,\beta)} \cdot \underbrace{\int_0^1 x^{a} \cdot (1-x)^{\beta-1} dx}_{B(a+1,\beta)} \xrightarrow[\text{of the Beta function}]{\text{Using the Property}}$$

$$E[X] = \frac{1}{B(a,\beta)} \cdot \frac{\Gamma(a+1)\Gamma(\beta)}{\Gamma(a+\beta+1)} \Longrightarrow$$

$$E[X] = \frac{\Gamma(a+\beta)}{\Gamma(a)\Gamma(\beta)} \cdot \frac{\Gamma(a+1)\Gamma(\beta)}{\Gamma(a+\beta+1)} \Longrightarrow$$

$$E[X] = \frac{\Gamma(a+\beta)\Gamma(a+1)}{\Gamma(a)\,\Gamma(a+\beta+1)} \Longrightarrow \begin{cases} \Gamma(a+1) = a\,\Gamma(a) \\ \Gamma(a+\beta+1) = (a+\beta)\Gamma(a+\beta) \end{cases}$$

$$E[X] = \frac{\Gamma(a+\beta)\, a\, \Gamma(a)}{\Gamma(a)(a+\beta)\Gamma(a+\beta)} \Rightarrow \boxed{E[X] = \frac{a}{a+\beta}}$$

(B): Calculating the variance, requires computing the second order moment $E[X^2]$:

$$E[X^2] = \int_0^1 x^2 \cdot f_X(x;\alpha,\beta)\,dx = \int_0^1 x^2 \cdot \frac{x^{\alpha-1}\cdot(1-x)^{\beta-1}}{B(\alpha,\beta)}\cdot dx \Rightarrow$$

$$E[X^2] = \frac{1}{B(\alpha,\beta)} \cdot \underbrace{\int_0^1 x^{\alpha+1}\cdot(1-x)^{\beta-1}\cdot dx}_{B(\alpha+2,\beta)} \Rightarrow$$

$$E[X^2] = \frac{1}{B(\alpha,\beta)} \cdot \frac{\Gamma(\alpha+2)\Gamma(\beta)}{\Gamma(\alpha+\beta+2)} \Rightarrow$$

$$E[X^2] = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \cdot \frac{\Gamma(\alpha+2)\Gamma(\beta)}{\Gamma(\alpha+\beta+2)} = \frac{\Gamma(\alpha+\beta)\Gamma(\alpha+2)}{\Gamma(\alpha)\Gamma(\alpha+\beta+2)} \Rightarrow$$

Utilizing the following identities

$$\begin{cases} \Gamma(\alpha+2) = (\alpha+1)\Gamma(\alpha+1) = (\alpha+1)\alpha\cdot\Gamma(\alpha) \\ \Gamma(\alpha+\beta+2) = (\alpha+\beta+1)\Gamma(\alpha+\beta+1) = (\alpha+\beta+1)(\alpha+\beta)\cdot\Gamma(\alpha+\beta) \end{cases}$$

$$E[X^2] = \frac{\Gamma(\alpha+\beta)\cdot\alpha\cdot(\alpha+1)\cdot\Gamma(\alpha)}{\Gamma(\alpha)\cdot(\alpha+\beta)\cdot(\alpha+\beta+1)} \Rightarrow$$

$$\boxed{E[X^2] = \frac{\alpha(\alpha+1)}{(\alpha+\beta)(\alpha+\beta+1)}}$$

Thus, we may write then:

$$Var[X] = E[X^2] - E^2[X] \Rightarrow$$

$$Var[X] = \frac{\alpha(\alpha+1)}{(\alpha+\beta)(\alpha+\beta+1)} - \frac{\alpha^2}{(\alpha+\beta)^2} \rightarrow$$

$$Var[X] = \frac{\alpha(\alpha+1)(\alpha+\beta)}{(\alpha+\beta)^2(\alpha+\beta+1)} - \frac{\alpha^2(\alpha+\beta+1)}{(\alpha+\beta)^2(\alpha+\beta+1)} \Rightarrow$$

$$Var[X] = \frac{(\alpha^2+\alpha)(\alpha+\beta)}{(\alpha+\beta)^2(\alpha+\beta+1)} - \frac{\alpha^2(\alpha+\beta+1)}{(\alpha+\beta)^2(\alpha+\beta+1)} \rightarrow$$

$$Var[X] = \frac{\alpha^3+\alpha^2\beta+\alpha^2+\alpha\beta-\alpha^3-\alpha^2\beta-\alpha^2}{(\alpha+\beta)^2(\alpha+\beta+1)} \rightarrow$$

$$\boxed{Var[X] = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}}$$

Derive the Moment Generating Function (MGF) for the Beta Distribution:

[#5]

By definition of the MGF, we have that:

$$M_X(t) = E[e^{tx}] = \int_0^1 e^{tx} \cdot \frac{x^{\alpha-1}(1-x)^{\beta-1}}{B(\alpha,\beta)} \cdot dx \Rightarrow$$

$$\boxed{M_X(t) = \frac{1}{B(\alpha,\beta)} \int_0^1 e^{tx} \cdot x^{\alpha-1} \cdot (1-x)^{\beta-1} \cdot dx \quad (1)}$$

We utilize the Taylor Series expansion of e^{tx}:

$$\boxed{e^{tx} = \sum_{k=0}^{\infty} \frac{(tx)^k}{k!} \quad (2)}$$

Combining Eqs (1) and (2), we get:

$$M_X(t) = \frac{1}{B(\alpha,\beta)} \cdot \int_0^1 \left[\sum_{k=0}^{\infty} \frac{(tx)^k}{k!} \right] \cdot \underbrace{x^{\alpha-1} \cdot (1-x)^{\beta-1}}_{\text{This product can be incorporated in the infinite sum}} dx \quad (3) \Rightarrow$$

$$\boxed{M_X(t) = \frac{1}{B(\alpha,\beta)} \cdot \int_0^1 \left[\sum_{k=0}^{\infty} \frac{(tx)^k}{k!} \cdot x^{\alpha-1} \cdot (1-x)^{\beta-1} \right] dx \quad (4)}$$

We can interchange the integration and summation operations as:

$$M_X(t) = \frac{1}{B(\alpha,\beta)} \cdot \sum_{K=0}^{\infty} \left[\int_0^1 \frac{(tx)^K}{K!} \cdot x^{\alpha-1} \cdot (1-x)^{\beta-1} \cdot dx \right] \quad (5)$$

Variables t and K may be treated as constants during the integration operation

$$M_X(t) = \frac{1}{B(\alpha,\beta)} \cdot \sum_{K=0}^{\infty} \left[\frac{t^K}{K!} \int_0^1 x^{K+\alpha-1} \cdot (1-x)^{\beta-1} \cdot dx \right] \quad (6)$$

Utilize the definition of the Beta Function

$$M_X(t) = \frac{1}{B(\alpha,\beta)} \cdot \sum_{K=0}^{\infty} \left[\frac{t^K}{K!} \cdot B(\alpha+K,\beta) \right] \quad (7)$$

This constant term can be moved inside the summation operation

$$M_X(t) = \sum_{K=0}^{\infty} \frac{t^K}{K!} \cdot \frac{B(\alpha+K,\beta)}{B(\alpha,\beta)} \quad (8)$$

Let this term be denoted as q_K.

$$q_K = \frac{B(\alpha+K,\beta)}{B(\alpha,\beta)} = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \cdot \frac{\Gamma(\alpha+K)\Gamma(\beta)}{\Gamma(\alpha+\beta+K)} \rightarrow$$

$$q_K = \underbrace{\frac{\Gamma(\alpha+K)}{\Gamma(\alpha)}}_{r_K} \cdot \underbrace{\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha+\beta+K)}}_{s_K} \quad (9)$$

Computing the first ratio $\underset{r_k}{\underset{\parallel}{\frac{\Gamma(a+k)}{\Gamma(a)}}}$ may be conducted as:

$$\Gamma(a+1) = \underset{\sim}{a}\,\Gamma(a)$$

$$\Gamma(a+2) = (a+1)\Gamma(a+1) = \underline{(a+1)a}\,\Gamma(a)$$

$$\Gamma(a+3) = (a+2)\Gamma(a+2) = \underline{(a+2)(a+1)a}\,\Gamma(a)$$

$$\vdots$$

$$\boxed{\Gamma(a+k) = (a+k-1)(a+k-2)\cdots(a+1)\cdot a\cdot\Gamma(a)\,,\quad k\geq 1 \quad (10)}$$

Thus, we have that $\boxed{r_k = \frac{\Gamma(a+k)}{\Gamma(a)} = (a+k-1)(a+k-2)\cdots(a+1)\cdot a \quad (11) \quad k\geq 1}$

Eq. (11) may be more compactly expressed as:

$$\boxed{r_k = \prod_{m=0}^{k-1}(a+m) \quad (12) \quad k\geq 1}$$

Likewise, for the inverse ratio $\frac{1}{s_k} = \frac{\Gamma(a+\beta+k)}{\Gamma(a+\beta)}$, we may write that:

$$\boxed{\frac{1}{s_k} = \prod_{m=0}^{k-1}(a+\beta+m) \quad (13) \quad k\geq 1}$$

Combining Eqs. (12) and (13), yields:

$$\frac{r_k}{s_k} = \frac{\prod_{m=0}^{k-1}(a+m)}{\prod_{m=0}^{k-1}(a+\beta+m)} \rightarrow \boxed{\frac{r_k}{s_k} = \prod_{m=0}^{k-1}\frac{a+m}{a+\beta+m} \quad (14) \quad k\geq 1}$$

Eq. (8) may be re-written as:

$$\boxed{M_X(t) = \sum_{K=0}^{\infty} \frac{t^K}{K!} \cdot q_K \quad (15)}$$

S.O.S: However, our expression for q_K according to Eq. (14) is defined only for values of K for which $K \geq 1$.

Therefore, we need to split the sum in Eq. (15) as:

$$M_X(t) = \frac{t^0}{0!} \cdot q_0 + \sum_{K=1}^{\infty} \frac{t^K}{K!} \cdot q_K \quad (16)$$

where $q_0 = \frac{B(\alpha+0,\beta)}{B(\alpha,\beta)} = \frac{B(\alpha,\beta)}{B(\alpha,\beta)} = 1 \quad (17)$

We may, thus, write:

$$\boxed{M_X(t) = 1 + \sum_{K=1}^{\infty} \frac{t^K}{K!} \cdot q_K \quad (18)}$$

Substituting Eq. (14) into Eq. (18) gives:

$$\boxed{M_X(t) = 1 + \sum_{K=1}^{\infty} \frac{t^K}{K!} \prod_{m=0}^{K-1} \frac{(\alpha+m)}{(\alpha+\beta+m)} \quad (19)}$$

The first and second degree moments of the Beta Distribution can also be derived by:

Eq. (19) can be re-written by changing the summation and product indices as:

$$M_X(t) = 1 + \sum_{n=1}^{\infty} \frac{t^n}{n!} \cdot \prod_{k=0}^{n-1} \frac{(a+k)}{(a+\beta+k)} \qquad (20)$$

We have also shown that the general form of the MGF is given by:

$$M_X(t) = \sum_{n=0}^{\infty} \frac{t^n}{n!} \cdot m_n = m_0 + \sum_{n=1}^{\infty} \frac{t^n}{n!} \cdot m_n \qquad (21)$$

From Eqs. (20) and (21), we can deduce that:

$$\begin{cases} m_0 = 1 \\ m_n = \prod_{k=0}^{n-1} \frac{(a+k)}{(a+\beta+n)}, \quad n \geq 1 \end{cases} \qquad (22)$$

From Eq. (22), we have that:

$$\begin{cases} m_1 = E[X] = \frac{a}{a+\beta} \quad \checkmark \\ m_2 = E[X^2] = \frac{a}{a+\beta} \cdot \frac{a+1}{a+\beta+1} \quad \checkmark \end{cases}$$