Notes on Probability and Statistics [LECTURE 1]
Continued

#33

Expected Values: The expected value of a random variable is merely its average value, where we speak of "average" value as one that is weighted according to the probability distribution.

The expected value of a distribution can be thought of as a measure of center, as we think of averages as being middle values.

By weighting the values of the random variable according to the probability distribution, we hope to obtain a number that summarizes a typical or expected value of an observation of the random variable.

★ The expected value or mean of a random variable $g(X)$ {X: is assumed to be a random variable}, denoted as $E[g(X)]$, is defined by:

$$E[g(x)] = \begin{cases} \boxed{\int_{-\infty}^{+\infty} g(x) f_X(x)\, dx}, & \boxed{\text{if } X \text{ is continuous;}} \\ \boxed{\sum_{x \in \mathcal{X}} g(x) f_X(x) = \sum_{x \in \mathcal{X}} g(x) P(X=x), \text{ if } X \text{ is discrete.}} \end{cases}$$

where $f_X(x)$ is probability density function when X is continuous or the probability mass function when X is discrete.

Example 1: Suppose that X has an exponential distribution given by:

$$f_X(x) = \frac{1}{\lambda} e^{-\frac{x}{\lambda}}, \quad \boxed{0 \le x < +\infty}, \quad \lambda > 0$$

Domain of the random variable.

Find the expected value $E[X]$.

Solution 1: X is apparently a continuous random variable. Thus, we can write that:

$$E[X] = \int_0^{+\infty} x \cdot \frac{1}{\lambda} e^{-\frac{x}{\lambda}} \, dx \Rightarrow$$

$$E[X] = \int_0^{+\infty} \frac{1}{\lambda} \cdot x \left(-\lambda e^{-\frac{x}{\lambda}}\right)' dx = \int_0^{+\infty} -\frac{1}{\lambda} \cdot \lambda \cdot x \left(e^{-\frac{x}{\lambda}}\right)' dx \Rightarrow$$

$$E[X] = \int_0^{+\infty} -x \cdot \left(e^{-\frac{x}{\lambda}}\right)' dx \xrightarrow[\text{ΚΑΤΑ ΠΑΡΑΓΟΝΤΕΣ}]{\text{ΟΛΟΚΛΗΡΩΣΗ}}$$

$$E[x] = \left[-x e^{-\frac{x}{\lambda}}\right]_0^{+\infty} - \int_0^{+\infty} -x' \cdot e^{-\frac{x}{\lambda}} \, dx \Rightarrow$$

$$E[X] = \underbrace{\left[-x e^{-\frac{x}{\lambda}}\right]_0^{+\infty}}_{[-0-0]} + \int_0^{+\infty} e^{-\frac{x}{\lambda}} \, dx \quad (A)$$

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★ Firstly, we need to compute the limit:

$$\lim_{x \to +\infty} x \cdot e^{-\frac{x}{\lambda}} \overset{(+\infty)\cdot(0)}{=\!=} \lim_{x \to +\infty} \frac{x}{e^{\frac{x}{\lambda}}} \overset{\left(\frac{\infty}{\infty}\right) \text{ De L'Hopital}}{=\!=}$$

$$\lim_{x \to +\infty} \frac{x'}{\frac{1}{\lambda} e^{x/\lambda}} = \lim_{x \to +\infty} \lambda \cdot \frac{1}{e^{x/\lambda}} = \lim_{x \to +\infty} \lambda \cdot e^{-\frac{x}{\lambda}} = 0 \quad (B)$$

★ Από (Α), (Β) έχουμε ότι: $E[X] = \int_0^{+\infty} e^{-\frac{x}{\lambda}} dx \Rightarrow$

$$E[X] = \int_0^{+\infty} (-\lambda \cdot e^{-\frac{x}{\lambda}})' dx \Rightarrow$$

$$E[X] = \left[-\lambda \cdot e^{-\frac{x}{\lambda}}\right]_0^{\infty} = \left[0 - (-\lambda)\cdot 1\right] = \lambda \Rightarrow$$

$$\boxed{E[X] = \lambda}$$

Example 2: Suppose that X has a binomial distribution (pmf) given by:

$$P(X=x) = \binom{n}{x} p^{x} \cdot (1-p)^{n-x}$$
$$x = 0, 1, 2, \dots, n.$$

where n is a positive integer and $0 \le p \le 1$. For every fixed pair n and p, the pmf sums to 1. Find the expected value $E[X]$.

Solution 2: Apparently, X is a discrete random variable.

Thus, we may write that:

$$E[X] = \sum_{x=0}^{x=n} x \cdot \binom{n}{x} \cdot p^{x} \cdot (1-p)^{n-x} = \sum_{x=1}^{x=n} x \cdot \binom{n}{x} p^{x} (1-p)^{n-x} \quad (A)$$

★ At this point, we are going to utilize the following identity:

$$\boxed{x \cdot \binom{n}{x} = n \cdot \binom{n-1}{x-1}} \quad (B)$$

We have that: $\binom{n}{x} = \frac{n!}{x!(n-x)!}$ and $\binom{n-1}{x-1} = \frac{(n-1)!}{(x-1)!(n-x)!}$

Thus, $x \cdot \binom{n}{x} = \frac{x \cdot n!}{x!(n-x)!} = \frac{n!}{(x-1)!(n-x)!}$

#36

Likewise, $n\binom{n-1}{x-1} = \frac{n(n-1)!}{(x-1)!(n-x)!} = \frac{n!}{(x-1)!(n-x)!}$

Utilizing Eq.(B), Eq.(A) may be written in the following form:

$$E[X] = \sum_{x=1}^{x=n} n\binom{n-1}{x-1} \cdot p^{x} \cdot (1-p)^{n-x} \xrightarrow[x=y+1]{\text{set } y=x-1}$$

$$E[X] = \sum_{y=0}^{y=n-1} n\binom{n-1}{y} \cdot p^{y+1} \cdot (1-p)^{n-(y+1)} \rightarrow$$

$$E[X] = np \boxed{\sum_{y=0}^{y=n-1} \binom{n-1}{y} p^{y} \cdot (1-p)^{(n-1)-y}} = 1 \rightarrow$$

This is the pmf of a discrete random variable Y which takes the values $y = \{0, 1, \dots, n-1\}$ and the sum is equal to 1.

$$\boxed{E[X] = np}$$

Example 3: A classic example of a random variable whose expected value does not exist is a Cauchy random variable whose pdf is given by:

$$\boxed{f_X(x) = \frac{1}{\pi} \cdot \frac{1}{1+x^2}, \quad \boxed{-\infty < x < +\infty}}$$

Domain of the Random variable

#37

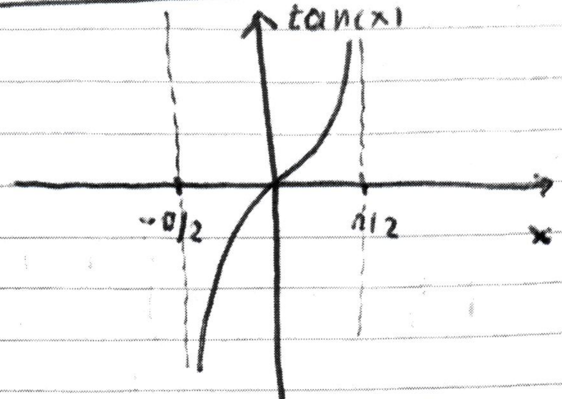
Solution 3: Initially, we would like to prove that the given $f_X(x)$ is a valid pdf. To do so, we must prove that:

$$\boxed{\int_{-\infty}^{+\infty} f_X(x)\,dx = 1 \quad (A)}$$

- We must remember that:

(*) $\lim_{x \to +\pi/2} \tan(x) = +\infty$

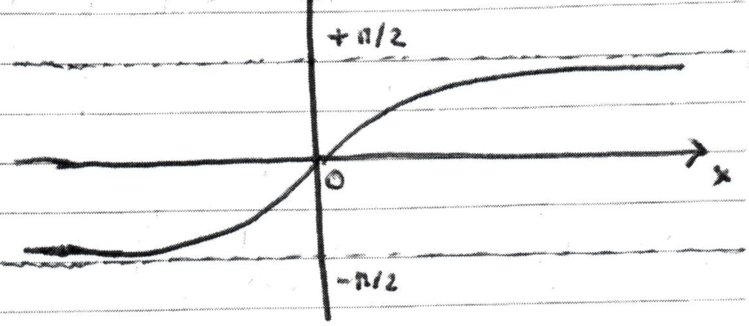
(*) $\lim_{x \to -\pi/2} \tan(x) = -\infty$



- By considering the inverse trigonometric function $\arctan(x)$, we get that:

(*) $\lim_{x \to +\infty} \arctan(x) = +\pi/2$

(*) $\lim_{x \to -\infty} \arctan(x) = -\pi/2$



- We must also remember that:

$$\boxed{\int \frac{1}{1+x^2} \cdot dx = \arctan(x) + c \quad (B)}$$

- Thus, we can write that:

$$\int_{-\infty}^{+\infty} \frac{1}{\pi} \cdot \frac{1}{1+x^2} \cdot dx = \frac{1}{\pi} \cdot \int_{-\infty}^{+\infty} (\arctan(x))'\,dx =$$

$$= \frac{1}{\pi} \cdot \Big[\arctan(x)\Big]_{-\infty}^{+\infty} = \frac{1}{\pi} \cdot \left[\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right] = \frac{1}{\pi} \cdot \pi = 1.$$

$\boxed{\#38}$

(★) Computing the expectation of the random variable x may be conducted as:

$$E[X] = \int_{-\infty}^{+\infty} x \cdot \frac{1}{\pi} \cdot \frac{dx}{1+x^2} = \boxed{\underbrace{\int_{-\infty}^{0} \frac{x\,dx}{\pi(1+x^2)}}_{I_A}} + \boxed{\underbrace{\int_{0}^{+\infty} \frac{x\,dx}{\pi(1+x^2)}}_{I_B}} \quad (\Gamma)$$

(★) At this point we need to evaluate the indefinite integral:

$$I = \int \frac{x\,dx}{\pi(1+x^2)} \;\overset{x^2+1=u}{\underset{du=2x\,dx}{=}}\; \int \frac{1/2\,du}{\pi u} = \frac{1}{2\pi}\int \frac{du}{u} = \frac{1}{2\pi}\ln|u| + C \Rightarrow$$

$$\boxed{I = \frac{1}{2\pi}\ln|x^2+1| + C} \quad (\Delta)$$

Apparently, none of the integrals I_A and I_B converge!!! (★)

(★) Eqs. (Γ) and (Δ) yield:

$$E[X] = \left[\frac{1}{2\pi}\ln|1+x^2|\right]_{-\infty}^{0} + \left[\frac{1}{2\pi}\ln|1+x^2|\right]_{0}^{+\infty} \quad (E)$$

• Due to the absolute value:

$$\lim_{x\to+\infty} \frac{1}{2\pi}\ln|x^2+1| = \lim_{x\to-\infty}\frac{1}{2\pi}\ln|x^2+1| = +\infty \quad (Z)$$

(★) Expressing integrals I_A and I_B as:

$$I_A = \lim_{R\to-\infty}\int_R^0 \frac{x\,dx}{\pi(1+x^2)} \quad \text{and} \quad I_B = \lim_{R\to+\infty}\int_0^R \frac{x\,dx}{\pi(1+x^2)}$$

we have that $\lim_{R\to-\infty} I_A(R) = (0-(+\infty)) = -\infty$

$$\lim_{R\to+\infty} I_B(R) = (+\infty - 0) = +\infty$$

Example 3: (Distance Minimization)

The expected value of a random variable may be thought of as relating to the interpretation of $E[X]$ as a good guess at a value of X.

★ Suppose that we want to measure the distance between a random variable and a constant b by $(X-b)^2$. The closer b is to X, the smaller this quantity is. We can determine the value of b by minimizing the quantity $E[(X-b)^2]$, and hence acquiring a good predictor for X. $\underbrace{}_{(\varnothing)}$

Solution 3: $E[(X-b)^2] = E[(X-E[X]+E[X]-b)^2] =$

$$= E[((X-E[X])+(E[X]-b))^2] \underline{\underline{E[\cdot] \text{ is a linear operator}}}$$

$E[a g_1(X) + \beta g_2(X) + c] =$
$E[a g_1(X)] + E[\beta g_2(X)] + E[c] =$
$a E[g_1(X)] + \beta E[g_2(X)] + c$

$$= E[(X-E[X])^2] + \underbrace{E[(E[X]-b)^2]}_{\text{CONSTANT VALUE}} + 2E[(X-E[X])(E[X]-b)] =$$

$$= E[(X-E[X])^2] + (E[X]-b)^2 + 2E[(X-E[X])(E[X]-b)]$$

★ Notice that: $E[(X-E[X])(\underbrace{E[X]-b}_{\text{CONSTANT VALUE}})] = (E[X]-b) \cdot E[(X-E[X])]$

★ Also, notice that: $E[(X-E[X])] = E[X] - E[E[X]] =$
$E[X] - E[X] = 0$.

★ Therefore, we can write that:

$$\boxed{\underbrace{E[(X-b)^2]}_{Q_0} = \underbrace{E[(X-E[X])^2]}_{Q_1} + \underbrace{(E[X]-b)^2}_{Q_2} \quad (A)}$$

#40

- Since we have no control over quantity Q_1, the only possibility for minimizing Q_0 is to set:

$$\boxed{b = E[X] \quad (B)}$$

- Thus, a good estimate for the random variable X is its expected value $E[X]$ and the respective quantity $E[(X-E[X])^2]$ is the variance.

ALTERNATIVE SOLUTION:

Formulate the cost functional w.r.t to the unknown parameter b:

$$\boxed{J(b) = E[(X-E[X])^2] + (E[X]-b)^2}$$

Solve the following minimization problem:

$$\boxed{\min_{b \in \mathbb{R}} J(b)}$$

$J(b)$: is quadratic w.r.t b and therefore possesses a minimum.

$\partial^2 J / \partial b^2 = 2 > 0 \Rightarrow$ LOCAL MINIMUM.

IMPOSE F.O.Cs:

$$\frac{\partial J}{\partial b} = 0 \Rightarrow \frac{\partial}{\partial b}\left\{E[(X-E[X])^2] + (E[X]-b)^2\right\} = 0 \Rightarrow$$

$$\frac{\partial}{\partial b}\left\{(E[X]-b)^2\right\} = 0 \Rightarrow 2(E[X]-b)\frac{\partial}{\partial b}\left\{E[X]-b\right\} = 0 \Rightarrow$$

$$-2(E[X]-b) = 0 \Rightarrow E[X]-b = 0 \Rightarrow \boxed{b^* = E[X]}$$