① **Properties of Expected Value:** { $E[\cdot]$: is a linear operator }

$$\boxed{\begin{aligned} &(\alpha)\ E[c] = c \\ &(\beta)\ E[cX] = cE[X] \\ &(\gamma)\ E[\alpha X + \beta Y] = \alpha E[X] + \beta E[Y] \end{aligned}}$$

② **Properties of Variance:**

(i) $\boxed{Var[X] = E[(X - E[X])^2]}$

(ii) $\boxed{Var[X] = Cov[X, X]}$

(iii) For two jointly distributed real-valued random variables X and Y with finite second moments $\{E[X^2] < \infty, E[Y^2] < \infty\}$, we have that:

$$\boxed{Cov[X,Y] = E[(X - E[X])(Y - E[Y])]}$$

(*) Derivation:

$$Cov[X,Y] = E[(X - E[X])(Y - E[Y])] \Rightarrow$$

$$Cov[X,Y] = E[X\cdot Y - X\cdot E[Y] - Y\cdot E[X] + E[X]E[Y]] \Rightarrow$$

$$Cov[X,Y] = E[X\cdot Y] - E[X\cdot E[Y]] - E[Y\cdot E[X]] + E[E[X]E[Y]] \Rightarrow$$

$$Cov[X,Y] = E[X\cdot Y] - E[X]E[Y] - \cancel{E[X]E[Y]} + \cancel{E[X]E[Y]} \Rightarrow$$

$$\boxed{Cov[X,Y] = E[X\cdot Y] - E[X]E[Y]}$$

(iv) Apparently, by setting $X = Y$, we can get that:

$$\boxed{Cov[X,X] = Var[X] = E[X^2] - E[X]^2}$$

(v) $Var[X] \geq 0$ (Variance is non-negative)

(vi) $Var[c] = 0$ (Variance of a constant is zero)

(vii) $Var[X] = 0 \iff \exists\, a \in \mathbb{R} : P(X=a) = 1$

(viii) $Var[a+X] = Var[X]$

(ix) $Var[aX] = a^2 Var[X]$

(*): Proof of Property ix

$$Var[aX] = E[a^2X^2] - E[aX]^2 \Rightarrow$$

$$Var[aX] = a^2E[X^2] - (aE[X])^2 \Rightarrow$$

$$Var[aX] = a^2E[X^2] - a^2E[X]^2 \Rightarrow$$

$$Var[aX] = a^2\{E[X^2] - E[X]^2\} \Rightarrow$$

$$Var[aX] = a^2 Var[X]$$

(x) Variance of sum of random variables:

$$Var[aX+\beta Y] = a^2Var[X] + \beta^2 Var[Y] + 2a\beta\, Cov[X,Y]$$

$$Var[aX-\beta Y] = a^2Var[X] + \beta^2 Var[Y] - 2a\beta\, Cov[X,Y]$$

$$Var[aX+\beta Y] = E[((aX+\beta Y) - E[aX+\beta Y])^2] \text{ (By Definition)} \Rightarrow$$

$$Var[aX+\beta Y] = E[((aX+\beta Y) - (E[aX]+E[\beta Y]))^2] \Rightarrow$$

$$Var[aX+\beta Y] = E[((aX+\beta Y) - (aE[X]+\beta E[Y]))^2] \Rightarrow$$

$$Var[aX+\beta Y] = E[(a(X-E[X]) + \beta(Y-E[Y]))^2] \Rightarrow$$

$$Var[aX+\beta Y] = E[a^2(X-E[X])^2 + \beta^2(Y-E[Y])^2 + 2a\beta(X-E[X])(Y-E[Y])] \Rightarrow$$

$$Var[aX+\beta Y] = a^2E[(X-E[X])^2] + \beta^2E[(Y-E[Y])^2] + 2a\beta\, E[(X-E[X])(Y-E[Y])] \Rightarrow$$

$$\boxed{Var[aX+\beta Y] = a^2Var[X] + \beta^2Var[Y] + 2a\beta\, Cov[X,Y]}$$

(xi) Linear Combinations:

$$\boxed{Var\left[\sum_{i=1}^{n} x_i\right] = \sum_{i=1}^{n}\sum_{j=1}^{n} Cov[x_i, x_j] + \sum_{i \neq j} Cov[x_i, x_j]}$$

or in its more general form:

$$\boxed{\begin{aligned} Var\left[\sum_{i=1}^{n} a_i x_i\right] &= \sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j Cov[x_i, x_j] = \\ &= \sum_{i=1}^{n} a_i^2 Var[x_i] + \sum_{i \neq j} a_i a_j Cov[x_i, x_j] \end{aligned}}$$

(*) Apparently, when the random variables X_i are pairwise independent, such that: $\boxed{Cov[x_i, x_j] = 0, \ \forall i \neq j}$

we can write that: $\boxed{Var\left[\sum_{i=1}^{n} x_i\right] = \sum_{i=1}^{n} Var[x_i]}$

▶ **Άσκηση**: Αν $(x_1, x_2, \ldots, x_n)$ τυχαίο δείγμα από άπειρο ή πεπερασμένο πληθυσμό με μέσο (μ) και διακύμανση (σ^2) όπου τα x_i είναι ανεξάρτητα τότε:

(i): $E[\bar{X}] = \frac{1}{n} E[x_1 + x_2 + \cdots + x_n] = \mu$

(ii): $Var[\bar{X}] = \frac{1}{n^2} Var[x_1 + x_2 + \cdots + x_n] \overset{iid}{=\!=} \frac{\sigma^2}{n}$

▶ **Λύση**: Για το (i) έχουμε ότι: $\boxed{\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i}$ και $\boxed{E[x_i] = \mu, \ \forall i \in [n]}$.

Επομένως, έχουμε ότι: $E[\bar{X}] = E[\frac{1}{n}\sum_{i=1}^{n} x_i] = \frac{1}{n} E[\sum_{i=1}^{n} x_i] =$

$$= \frac{1}{n}\sum_{i=1}^{n} E[x_i] = \frac{1}{n}\cdot n \cdot \mu = \mu$$

Για το (ii) έχουμε ότι: $\boxed{Var[x_i] = \sigma^2, \ \forall i \in [n]}$

$$Var[\bar{x}] = Var[\frac{1}{n}\sum_{i=1}^{n} x_i] = \frac{1}{n^2} Var[\sum_{i=1}^{n} x_i] = \frac{1}{n^2}\sum_{i=1}^{n} Var[x_i] \Rightarrow$$

$$Var[\bar{x}] = \frac{1}{n^2}\cdot n \cdot \sigma^2 = \frac{\sigma^2}{n}$$

#5

Problem: A poker hand (5 cards) is dealt off the top of a well-shuffled deck of 52 cards.
Let X be the number of diamonds in the hand.
Let Y be the number of hearts in the hand.

(α): Do you think $Cov[X,Y]$ is positive, negative or zero.

(β): Let D_i $(i=1,2,3,4,5)$ be a random variable that is 1 if the ith card is a diamond and 0 otherwise. What is $E[D_i]$?

(γ): Let H_i $(i=1,2,3,4,5)$ be a random variable that is 1 if the i-th card is a heart and 0 otherwise. Prove that $E[H_i]$ is the same as $E[D_i]$. What is $Cov[D_i,H_i]$? What is $Cov[D_i,H_j]$?, when $i \neq j$. Keep in mind that D_i and H_i are indicator random variables that take on values 0 or 1.
[Hint: make a table for the joint p.m.f. There are only 4 possible outcomes]

(δ): Use the previous answers on questions (β),(γ) to calculate $Cov[X,Y]$?