# ⊛ Correlation / Covariance Formula

- The correlation of X and Y is the number defined by:

$$\boxed{\rho_{xy} = \frac{Cov(X,Y)}{\sigma_x \sigma_y}}$$

- Prove that $\boxed{Var[aX+\beta Y] = a^2 Var[X] + \beta^2 Var[Y] + 2a\beta Cov[X,Y]}$

⊛ The mean of $aX+\beta Y$ is $E[aX+\beta Y] = aE[X]+\beta E[Y] \Rightarrow$

$$\boxed{E[aX+\beta Y] = a\mu_x + \beta\mu_y}$$

Thus, we may write that:

$$Var[aX+\beta Y] = E\left[\left((aX+\beta Y) - E[aX+\beta Y]\right)^2\right] =$$

$$= E\left[\left((aX+\beta Y) - (a\mu_x + \beta\mu_y)\right)^2\right] =$$

$$= E\left[\left\{a(X-\mu_x) + \beta(Y-\mu_y)\right\}^2\right] =$$

$$= E\left[a^2(X-\mu_x)^2 + \beta^2(Y-\mu_y)^2 + 2a\beta(X-\mu_x)(Y-\mu_y)\right]$$

$$= a^2 E\left[(X-\mu_x)^2\right] + \beta^2 E\left[(Y-\mu_y)^2\right] +$$

$$+ 2a\beta E\left[(X-\mu_x)(Y-\mu_y)\right] =$$

$$= a^2 Var[X] + \beta^2 Var[Y] + 2a\beta\, Cov[X,Y]$$

(*) For any random variables $X$ and $Y$,

a. $-1 \leq \rho_{xy} \leq +1$

β. $|\rho_{xy}| = 1$ if and only if there exist numbers $a \neq 0$ and $b$, such that $\underline{P}(Y = aX + \beta) = 1$.

If $\rho_{xy} = 1$, then $a > 0$.

If $\rho_{xy} = -1$, then $a < 0$.

- Consider the expression:

$$h(t) = E\left[\left\{(X-\mu_x)t + (Y-\mu_y)\right\}^2\right] \Rightarrow$$

$$h(t) = t^2 E[(X-\mu_x)^2] + 2t E[(X-\mu_x)(Y-\mu_y)] + E[(Y-\mu_y)^2] \Rightarrow$$

$$\boxed{h(t) = t^2 \cdot \sigma_x^2 + 2t\,Cov[X,Y] + \sigma_y^2}$$

- The quadratic function $h(t)$ corresponds to the expected value of a non-negative random variable, and, therefore, is greater than or equal to $0$ for any possible value of $t$.

- $\boxed{\Delta h = 4\,Cov[X,Y]^2 - 4\sigma_x^2\sigma_y^2 \quad (\text{Discriminant of } h(t))}$

- Thus, $h(t)$ must have at least one positive root which is satisfied when: $\Delta h \leq 0 \rightarrow 4 \cdot (Cov[X,Y]^2 - \sigma_x^2\sigma_y^2) \leq 0 \Rightarrow$

$$\Delta h = 4 \cdot (Cov[X,Y] - \sigma_x\sigma_y)(Cov[X,Y] + \sigma_x\sigma_y) \leq 0$$

| | | $-\sigma_x\sigma_y$ | | $0$ | | $+\sigma_x\sigma_y$ | |
|---|---|---|---|---|---|---|---|
| $\Delta h$ | $+$ | | $-$ | | $-$ | | $+$ |

$$-\sigma_x\sigma_y \leq Cov[X,Y] \leq +\sigma_x\sigma_y \Rightarrow$$

$$-1 \leq \underline{Cov[X,Y]} \leq +1 \Rightarrow \boxed{-1 \leq \rho_{xy} \leq +1}$$

- The sign of the discriminant function $\Delta h'$ can take the form $\Delta h' = \dfrac{\Delta h}{4\,\delta_x^2 \delta_y^2} \Rightarrow$ $\boxed{\text{sign}[\Delta h] = \text{sign}[\Delta h']}$

$$\boxed{\Delta h' = (\rho_{xy} - 1)(\rho_{xy} + 1)}$$

- Therefore, when $|\rho_{xy}| = 1 \Rightarrow \boxed{\Delta h' = 0}$.
- That is, $|\rho_{xy}| = 1$ if and only if $h(t)$ has a single root.
- But since $\{(X-\mu_x)t + (Y-\mu_y)\}^2 \geq 0$,

the expected value $h(t) = E[\{(X-\mu_x)t + (Y-\mu_y)\}^2] = 0$, if and only if

$$P[\{(X-\mu_x)t + (Y-\mu_y)\}^2 = 0] = 1 \Leftrightarrow$$

$$\boxed{P[(X-\mu_x)t + (Y-\mu_y) = 0] = 1}$$

- We have that $Xt^* - \mu_x t^* + Y - \mu_y = 0 \Rightarrow$

$$Y = -t^* X + t^* \mu_x + \mu_y \Rightarrow$$

$$\boxed{\begin{cases} Y = aX + \beta \\ a = -t^* \\ \beta = t^* \mu_x + \mu_y \end{cases}}$$

- This yields that $P[Y = aX + \beta] = 1$

- The unique root of the quadratic function will be given as:

$$t^* = -\frac{2\,Cov[X,Y] \pm \sqrt{\Delta u}}{2\sigma_x^2} = -\frac{Cov[X,Y]}{\sigma_x^2}$$

- In this case, we have that: $a = -t^* = \frac{Cov[X,Y]}{\sigma_x^2} \rightarrow$

$$sign[a] = sign\left[\frac{Cov[X,Y]}{\sigma_x^2}\right] = sign\left[Cov[X,Y]\right]$$

$$= sign[\rho_{XY}].$$