ΚΑΤΑΝΟΜΗ x^2

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#1

(Ι) Αν οι τ.μ. $X_1, X_2, \dots, X_n$ είναι ανεξάρτητες και ακολουθούν την τυποποιημένη κανονική κατανομή $X_i \sim N(0,1)$, $\forall i \in [n]$, τότε η τυχαία μεταβλητή $Y = X_1^2 + X_2^2 + \dots + X_n^2$ ακολουθεί την λεγόμενη κατανομή χι-τετράγωνο (chi-squared) με n βαθμούς ελευθερίας, η οποία συμβολίζεται με χ_n^2 και έχει συνάρτηση πυκνότητας πιθανότητας:

$$\boxed{f_X(x) = \begin{cases} \dfrac{1}{2^{n/2}\,\Gamma(n/2)} \cdot x^{n/2-1} \cdot e^{-x/2}, & x > 0; \\ 0, & x < 0. \end{cases}}$$

(*) Η συνάρτηση $\Gamma : \mathbb{R}^+ \to \mathbb{R}^+$ ορίζεται ως εξής:

$$\boxed{\Gamma(z) = \int_0^{\infty} x^{z-1} \cdot e^{-x}\, dx \quad (A)}$$

(*) <u>ΙΔΙΟΤΗΤΕΣ ΣΥΝΑΡΤΗΣΗΣ Γ:</u>

(Β): $\Gamma(z) = \Gamma(z-1)\cdot(z-1)$ (s.o.s)

(Γ): $\Gamma(n) = (n-1)!$ (s.o.s)

(Δ): $\Gamma(1/2) = \sqrt{\pi}$ (s.o.s)

(Ε): $\Gamma(n+\frac{1}{2}) = \sqrt{\pi} \cdot \prod_{j=0}^{n-1} (j+\frac{1}{2})$

Gamma Distribution: The Gamma Distribution can be parameterized in terms of a scale parameter (a) and an inverse scale parameter (β), also called rate parameter.

▶ $x \sim \Gamma(a, \beta) \equiv \text{Gamma}(a, \beta)$

▶ The corresponding probability density function in the shape-rate parameterization is given by:

$$\boxed{f_X(x) = \frac{x^{a-1} \cdot e^{-\beta x} \cdot \beta^{a}}{\Gamma(a)}}$$

Show that $z_1^2 + z_2^2 + \dots + z_n^2 \sim \chi_n^2$ where $z_i \sim N(0,1)$, $\forall\, i \in [n]$ with z_i independent.

Step 1: Let $\boxed{W = \sum_{k=1}^{n} z_k^2 \quad (I)}$ and try to formulate the moment generating function for the random variable W as:

$$M_W(t) = E[e^{Wt}] = E[e^{(z_1^2 + z_2^2 + \dots + z_n^2)t}] \Rightarrow$$

$$\boxed{M_W(t) = E[e^{z_1^2 t} \cdot e^{z_2^2 t} \cdot \dots \cdot e^{z_n^2 t}] \quad (II)}$$

We know that when U and V are independent random variables, it holds:

$$\boxed{E[U \cdot V] = E[U] \cdot E[V] \quad (III)}$$

which transforms Eq. (II) as:

$$\boxed{M_W(t) = E[e^{z_1^2 t}] \cdot E[e^{z_2^2 t}] \cdot \dots \cdot E[e^{z_n^2 t}] \quad (IV)}$$

Step 2: We know that: $\forall i \in [n]$, $z_i \sim N(0,1)$ or the corresponding probability density function will be given as:

$$\boxed{f(z_i) = \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{z_i^2}{2}} \quad (V)}$$

- For the k-th term appearing in Eq. (IV), we can write:

$$E[e^{z_k^2 t}] = \int_{-\infty}^{+\infty} e^{z_k^2 t} \cdot f(z_k)\, dz_k \rightarrow$$

$$E[e^{z_k^2 t}] = \int_{-\infty}^{+\infty} e^{z_k^2 t} \cdot \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{z_k^2}{2}} \cdot dz_k \rightarrow$$

$$\boxed{E[e^{z_k^2 t}] = \frac{1}{\sqrt{2\pi}} \cdot \int_{-\infty}^{+\infty} e^{-z_k^2(-t+\frac{1}{2})}\, dz_k \quad (VI)}$$

- Taking into account the expression for the Gaussian integral, we may write that:

$$\boxed{I = \int_{-\infty}^{+\infty} e^{-x^2} \cdot dx = \sqrt{\pi} \quad (VII)}$$

- Setting $x = z_k(-t+\frac{1}{2})^{1/2} \rightarrow dx = dz_k(-t+\frac{1}{2})^{1/2}$, we can re-express Eq. (VII) as:

$$\boxed{I = \int_{-\infty}^{+\infty} e^{-z_k^2(-t+1/2)} \cdot dz_k \cdot (-t+\frac{1}{2})^{1/2} \quad (VIII)}$$

- We may also write that: $z_k = \frac{x}{\sqrt{-t+1/2}}$, which yields that:

$$-t+\frac{1}{2} > 0 \Rightarrow \boxed{t < \frac{1}{2}.}$$

➢ Eqs. (VII) and (VIII) yield that:

$$\boxed{\int_{-\infty}^{+\infty} e^{-2z_k^2\left(-t+\frac{1}{2}\right)}\, dz_k = \frac{\sqrt{\pi}}{\left(-t+\frac{1}{2}\right)^{1/2}} \quad \text{for } t < 1/2.} \quad (IX)$$

➢ Eqs. (VI) and (IX) can be combined as:

$$\boxed{E\left[e^{z_k^2 t}\right] = \frac{1}{\sqrt{2\pi}} \cdot \frac{\sqrt{\pi}}{\left(-t+\frac{1}{2}\right)^{1/2}} = \frac{1}{(-2t+1)^{1/2}} = \frac{1}{(1-2t)^{1/2}} \quad (X)}$$

Step 3: The moment generating function for the random variable W, can be ultimately expressed as:

$$M_W(t) = E\left[e^{z_1^2 t}\right] \cdot E\left[e^{z_2^2 t}\right] \cdots E\left[e^{z_n^2 t}\right] \Rightarrow$$

$$\downarrow$$

$$(1-2t)^{-1/2} \cdot (1-2t)^{-1/2} \cdots (1-2t)^{-1/2}$$

$$M_W(t) = \left[\frac{1}{(1-2t)^{1/2}}\right]^n \Rightarrow \boxed{M_W(t) = \frac{1}{(1-2t)^{n/2}}} \quad (XI)$$

Step 4: Finally, we need to conduct a thorough computation of moment generating function for the χ_n^2 distribution.

➢ We know that if $X \sim \chi_n^2$ then the respective probability density function will be given as:

$$f_X(x) = \begin{cases} \dfrac{1}{2^{n/2}\Gamma(n/2)} \cdot e^{-\frac{x}{2}} \cdot x^{\frac{n}{2}-1}, & x > 0; \\ 0, & x < 0. \end{cases}$$ [1]

▸ In this context, we have that:

$$M_X(t) = E[e^{tX}] = \int_0^{\infty} e^{tx} \cdot f_X(x)\,dx \rightarrow$$

$$M_X(t) = \int_0^{\infty} e^{tx} \cdot \frac{1}{2^{n/2} \cdot \Gamma(n/2)} \cdot e^{-\frac{x}{2}} \cdot x^{\frac{n}{2}-1}\,dx \rightarrow$$

$$M_X(t) = \frac{1}{2^{n/2} \cdot \Gamma(n/2)} \cdot \int_0^{\infty} e^{tx} \cdot e^{-\frac{x}{2}} \cdot x^{\frac{n}{2}-1}\,dx \rightarrow$$

$$\boxed{M_X(t) = \frac{1}{2^{n/2} \cdot \Gamma(n/2)} \cdot \int_0^{\infty} x^{\frac{n}{2}-1} \cdot e^{x(t-\frac{1}{2})}\,dx \quad [2]}$$

Converges for $t < \frac{1}{2}$!!!

▸ By utilizing the following change of variables, we get that:

$$u = \left(\frac{1}{2} - t\right)x \text{ or } \boxed{u = -\left(t - \frac{1}{2}\right) \cdot x \quad [3]}$$

$$\boxed{du = \left(\frac{1}{2} - t\right)dx \quad [4]} \quad \text{and} \quad \boxed{x = \frac{u}{1/2 - t} \quad [5]}$$

$$\boxed{dx = \frac{du}{1/2 - t} \quad [6]}$$

Substituting Eqs. (3, 5, 6) into Eq. (2), we get that:

$$M_x(t) = \frac{1}{2^{n/2}\Gamma(n/2)} \cdot \int_0^{\infty} \frac{u^{n/2-1}}{\boxed{(1/2-t)^{n/2-1}}} \cdot e^{-u} \cdot \frac{du}{\boxed{(1/2-t)}} \Rightarrow$$

$$M_x(t) = \frac{1}{2^{n/2}\cdot\Gamma(n/2)} \cdot \int_0^{\infty} \frac{1}{(1/2-t)^{n/2}} \cdot u^{n/2-1} \cdot e^{-u} \cdot du \rightarrow$$

$$\boxed{M_x(t) = (1/2-t)^{-n/2} \cdot \frac{1}{2^{n/2}\Gamma(n/2)} \cdot \int_0^{\infty} u^{n/2-1} \cdot e^{-u} \cdot du \quad [7]} \Rightarrow$$

$$M_x(t) = \boxed{\left(\frac{1}{2}-t\right)^{-n/2} \cdot 2^{-n/2}} \cdot \frac{1}{\Gamma(n/2)} \cdot \int_0^{\infty} u^{n/2-1} \cdot e^{-u} du \rightarrow$$

$$M_x(t) = (1-2t)^{-n/2} \cdot \frac{1}{\Gamma(n/2)} \cdot \boxed{\underbrace{\int_0^{\infty} u^{n/2-1} \cdot e^{-u} du}_{\Gamma(n/2)}} \Rightarrow$$

We know that:

$$\Gamma(z) = \int_0^{\infty} x^{z-1} \cdot e^{-x} dx$$

Finally, we get that:

$$\boxed{M_x(t) = (1-2t)^{-n/2}} \quad \text{or}$$

$$\boxed{M_x(t) = \frac{1}{(1-2t)^{n/2}} \quad [8]}$$

Comparing Eqs. (8) and (x1), it is obvious that the random variable $W \sim \chi_n^2$.

II: Εάν $X \sim \chi^2_n$ τότε $\begin{cases} E[X] = n \\ Var[X] = 2n \end{cases}$

Γνωρίζουμε ότι: η τ.μ. X μπορεί να γραφεί στην μορφή

$$\boxed{X = Y_1^2 + Y_2^2 + \dots + Y_n^2 \quad \text{με} \quad Y_k \sim N(0,1), \; \forall k \in [n]} \quad \Gamma_1$$

Y_k's INDEPENDENT!!!

- Γνωρίζουμε ότι: $\boxed{Var[Y_k] = E[Y_k^2] - E^2[Y_k], \; \forall k \in [n]}$ Γ_2

- Γνωρίζουμε επιπλέον ότι: $\begin{cases} \forall k \in [n], \; E[Y_k] = \\ \forall k \in [n], \; Var[Y_k] = 1 \end{cases}$ Γ_3

- Από τις σχέσεις Γ_2 και Γ_3 μπορούμε να εξάγουμε ότι:

$$E[Y_k^2] = \underbrace{Var[Y_k]}_{1} + \underbrace{E^2[Y_k]}_{0} \Rightarrow \boxed{E[Y_k^2] = 1 \quad \Gamma_4 \quad \forall k \in [n]}$$

- Επομένως, μπορούμε να γράψουμε ότι:

$$E[X] = E\left[\sum_{k=1}^{n} Y_k^2\right] = \sum_{k=1}^{n} E[Y_k^2] = \sum_{k=1}^{n} 1 = n \quad (\Gamma_5)$$

- Για την διακύμανση της τ.μ. X έχουμε ότι:

$$Var[X] = Var\left[\sum_{k=1}^{n} Y_k^2\right] \overset{\text{INDPND}}{=\!=} \sum_{k=1}^{n} 1^2 \cdot Var[Y_k^2] = \sum_{k=1}^{n} Var[Y_k^2] \quad (\Gamma_6)$$

- Με βάση την σχέση Γ_2 μπορούμε να γράψουμε ότι:

$$Var[Y_k^2] = E[Y_k^4] - E^2[Y_k^2], \; \forall k \in [n] \quad (\Gamma_7)$$

- Σύμφωνα με τα παραπάνω, μπορούμε να γράψουμε ότι:

$$Var[Y_k^2] = m_4 - m_2^2 \qquad (Γ8)$$

- Ωστόσο, έχουμε αποδείξει ότι: $m_{2n} = \frac{(2n)!}{n!\,2^n} \xrightarrow{n=2} m_4 = \frac{4!}{2!\,4} = \frac{24}{2\cdot 4}$

$$Var[Y_k^2] = 3 - 1 = 2 \qquad (Γ9)$$

$$\rightarrow \boxed{m_4 = \frac{24}{8} = 3}$$

- Τελικά, αποδείξαμε ότι: $Var[X] = \sum_{k=1}^{n} m_4 - m_2^2 = \sum_{k=1}^{n} 2 = 2n$ (Γ10)