Ν.δ.Ο αν η Τ.Μ. X ακολουθεί την τυποποιημένη κανονική κατανομή $N(0,1)$, τότε η Τ.Μ. X^2 ακολουθεί την κατανομή χ_1^2 με $(n=1)$ ένα βαθμό ελευθερίας.

- Let Z be a standard normal variable and let X be its square. Then, we have to demonstrate that

$$\boxed{X \sim \chi_1^2}$$

- Given that $X = Z^2$, we have that $X \geq 0$.

- **Problem Solving Strategy:**

  (a) Try to derive the cumulative distribution function of X, $F_X(x)$. (cdf)

  (β) Try to derive the probability distribution function of x, $f_X(x)$ (pdf)

- The cumulative distribution function or distribution function of X, is given by:

$$F_X(x) = P(X \leq x) \quad (1) \rightarrow$$

$$\boxed{F_X(x) = P(Z^2 \leq x) \quad (2)}$$

- At this point, we need to re-express inequality $z^2 \leq x$ in terms of z.

- We have that: $z^2 \leq x \Rightarrow z^2 - x \leq 0 \Rightarrow$

$$z^2 - (\sqrt{x})^2 \leq 0 \Rightarrow \boxed{(z - \sqrt{x})(z + \sqrt{x}) \leq 0}$$

- Thus, we have a second-degree polynomial with respect to z, such that: $\boxed{Q(z) = (z - \sqrt{x})(z + \sqrt{x})}$

- Determining the two roots of $Q(z)$, yields that:

$$\boxed{Q(z) = 0 \Rightarrow \begin{cases} z_+ = +\sqrt{x}, \\ z_- = -\sqrt{x} \end{cases} \quad (\text{Remember that } x \geq 0)}$$

- Therefore, we may write that:

z		$-\sqrt{x}$	0	$+\sqrt{x}$	
$Q(z)$	$+$	$\circ$	$-$	$\circ$	$+$

- By considering the sign of $Q(z)$ when $z = 0$, we can determine the sign of the polynomial between its roots, i.e. when $-\sqrt{x} < z < \sqrt{x}$. Thus, we have that $Q(0) = -\sqrt{x} \cdot \sqrt{x} = -x \leq 0$

- Therefore, demanding $Q(z) \leq 0$ is equivalent to demanding that

$$\boxed{-\sqrt{x} \leq z \leq +\sqrt{x}}$$

(*) In this context, Eq. (2) is equivalent to:

$$\boxed{F_X(x) = P\left(-\sqrt{x} \leq Z \leq +\sqrt{x}\right) \quad (3)}$$

(*) However, we know that the random variable Z is a standardized normal variable with the following probability distribution function:

$$\boxed{f_Z(z) = \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{z^2}{2}} \quad (4)}$$

(*) In this setting, Eq. (3) can be re-written as:

$$\boxed{F_X(x) = \int_{-\sqrt{x}}^{+\sqrt{x}} f_Z(z)\,dz \quad (5)}$$

(*) We also know that for $x < 0$, $F_X(x) = \varnothing$, since being a square cannot result in taking negative values.

(*) At this point, we will exploit the fact the density function of a random variable is the first derivative of its distribution function.

(*) Therefore, we may write that:

$$f_X(x) = \frac{d}{dx} F_X(x) = \frac{d}{dx} \int_{-\sqrt{x}}^{+\sqrt{x}} f_Z(z)\,dz \quad (6)$$

(*) At this point we are going to utilize Leibnitz's integral rule which states as follows:

LEIBNITZ'S INTEGRAL RULE:

If $f(x,\theta)$, $a(\theta)$ and $\beta(\theta)$ are differentiable with respect to θ, then the following holds:

$$\frac{d}{d\theta} \int_{a(\theta)}^{\beta(\theta)} f(x,\theta)\,dx = f(\beta(\theta),\theta) \cdot \frac{d}{d\theta}\beta(\theta) - f(a(\theta),\theta) \cdot \frac{d}{d\theta} a(\theta) + \int_{a(\theta)}^{\beta(\theta)} \frac{\partial}{\partial\theta} f(x,\theta)\,dx$$

Apparently, if $a(\theta)$ and $\beta(\theta)$ are constant (i.e. not functions of θ), the previous formula can be re-expressed as:

$$\frac{d}{d\theta} \int_{a}^{\beta} f(x,\theta)\,dx = \int_{a}^{\beta} \frac{\partial}{\partial\theta} f(x,\theta)\,dx$$

- Thus, in this particular case we have that:

$$f_X(x) = \underset{\theta}{\frac{d}{dx}} \int_{\underset{\alpha(\theta)}{-\sqrt{x}}}^{\overset{\beta(\theta)}{+\sqrt{x}}} f_Z(\cancel{z}) d\cancel{z}$$

- We also have that:

$$f(x,\theta) = f(x)$$

which yields that function $f(\cdot)$ does not depend on θ.

- In our case, function $f_Z(z)$ does not depend on x.

- A direct application of Leibnitz's integration rule gives:

$$f_X(x) = f_Z(\sqrt{x}) \frac{d}{dx}\sqrt{x} - f_Z(-\sqrt{x}) \frac{d}{dx}(-\sqrt{x}) + \int_{-\sqrt{x}}^{+\sqrt{x}} \underbrace{\frac{\partial}{\partial x} f_Z(z) dz}_{\emptyset} \rightarrow$$

$$f_X(x) = \frac{1}{\sqrt{2\pi}} \cdot \exp\left[-\frac{1}{2}(\sqrt{x})^2\right] \cdot \frac{1}{2\sqrt{x}} - \frac{1}{\sqrt{2\pi}} \cdot \exp\left[-\frac{1}{2}(-\sqrt{x})^2\right] \cdot \left(-\frac{1}{2\sqrt{x}}\right) \rightarrow$$

$$f_X(x) = \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{2} \cdot \frac{1}{\sqrt{x}} \cdot \exp\left[-\frac{1}{2}x\right] + \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{2} \cdot \frac{1}{\sqrt{x}} \cdot \exp\left[-\frac{1}{2}x\right] \rightarrow$$

$$f_X(x) = \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{\sqrt{x}} \cdot \exp\left[-\frac{1}{2}x\right] \rightarrow$$

$$f_X(x) = \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{\pi}} \cdot x^{-1/2} \cdot \exp\left[-\frac{1}{2}x\right] \rightarrow$$

$$f_X(x) = \frac{1}{2^{1/2} \cdot \Gamma(1/2)} \cdot x^{1/2-1} \cdot e^{-\frac{x}{2}} \quad (\text{since } \sqrt{\pi} = \Gamma(1/2))$$

- For $x < 0$, it is trivial to say that $f_X(x) = \emptyset$.

(*) Finally, we may write that:

$$f_X(x) = \begin{cases} \dfrac{1}{2^{1/2} \cdot \Gamma(1/2)} \cdot x^{\frac{1}{2}-1} \cdot e^{-\frac{x}{2}} & , \ x \geq 0 ; \\ \varnothing & , \ x \leq \varnothing . \end{cases}$$

(*) The acquired density function corresponds to the pdf of a chi-squared distribution with $n=1$ degrees of freedom.

(*) Note: The pdf for a random variable $X \sim \chi^2_v$ is given by:

$$f_X(x) = \begin{cases} \dfrac{1}{2^{v/2} \Gamma(v/2)} \cdot x^{\frac{v}{2}-1} \cdot e^{-\frac{x}{2}} & ; \ x \geq 0 \\ \varnothing & , \ x = 0 . \end{cases}$$