Average Risk Minimization

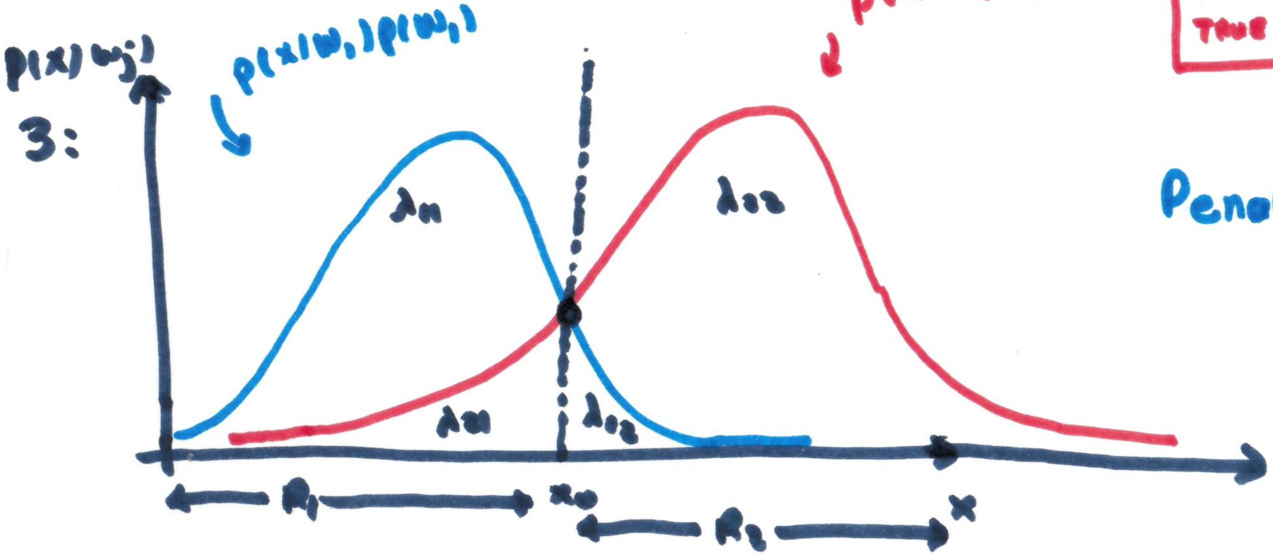
#1

1: Each decision is assigned with a penalty term.

2: λ_{ij} : penalty term when deciding in favor of ω_j when the true class is ω_i.

λ_{ij} — i: TRUE CLASS, j: DECIDED CLASS

3:



Penalty Matrix : $\underline{L} = \begin{bmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{21} & \lambda_{22} \end{bmatrix}$

4: Average Risk w.r.t. ω_1 :

Probability of wrong decisions in favor of ω_2

$$r_1 = \overbrace{\int_{R_1} \lambda_{11}\, p(x|\omega_1)\, p(\omega_1)\, dx}^{\text{AVERAGE RISK ON } R_1} + \overbrace{\boxed{\int_{R_2} \lambda_{12}\, p(x|\omega_1)\, p(\omega_1)\, dx}}^{\text{AVERAGE RISK ON } R_2} \quad (\mathrm{I})$$

5: Average Risk w.r.t. ω_2 :

$$r_2 = \boxed{\int_{R_1} \lambda_{21}\, p(x|\omega_2)\, p(\omega_2)\, dx} + \int_{R_2} \lambda_{22}\, p(x|\omega_2)\, p(\omega_2)\, dx \quad (\mathrm{II})$$

Probability of wrong decisions in favor of ω_1

6: Average Risk : $r = r_1 + r_2$ (III)

7: Average Risk on R_1:

$$\int_{R_1} \Big[\underbrace{\lambda_{11} p(x|\omega_1) p(\omega_1) + \lambda_{21} p(x|\omega_2) p(\omega_2)}_{\ell_1(x)} \Big] dx \quad (IV)$$

8: Average Risk on R_2:

$$\int_{R_2} \Big[\underbrace{\lambda_{12} p(x|\omega_1) p(\omega_1) + \lambda_{22} p(x|\omega_2) p(\omega_2)}_{\ell_2(x)} \Big] dx \quad (V)$$

9: **Modified Decision Rule:**

$$\boxed{g(x) = \begin{cases} \omega_1, & \ell_1(x) < \ell_2(x); \\ \omega_2, & \ell_2(x) < \ell_1(x). \end{cases}} \quad \text{(VI)}$$

10: **Modified Decision Boundary:** x_0^* : $\ell_1(x) = \ell_2(x) \Leftrightarrow$

$$\lambda_{11} p(x/w_1)p(w_1) + \lambda_{21} p(x/w_2)p(w_2) = \lambda_{12} p(x/w_1)p(w_1) + \lambda_{22} p(x/w_2)p(w_2) \Leftrightarrow$$

$$(\lambda_{11} - \lambda_{12})p(x/w_1)p(w_1) = (\lambda_{22} - \lambda_{21})p(x/w_2)p(w_2) \Leftrightarrow$$

$$\frac{p(x/w_1)}{p(x/w_2)} = \frac{p(w_2)}{p(w_1)} \cdot \frac{\lambda_{22} - \lambda_{21}}{\lambda_{11} - \lambda_{12}} \text{ or } \boxed{\frac{p(x/w_1)}{p(x/w_2)} = \frac{p(w_2)}{p(w_1)} \cdot \frac{\lambda_{21} - \lambda_{22}}{\lambda_{12} - \lambda_{11}}} \quad \text{(VII)}$$

#5

II: It is reasonable to assign zero risk when making the correct decision (that is, the penalty matrix is diagonal):

$$L = \begin{bmatrix} 0 & \lambda_{12} \\ \lambda_{21} & 0 \end{bmatrix} \qquad (\lambda_{11} = \lambda_{22} = 0)$$

Thus, Eq. VII will be written as:

$$\boxed{\frac{p(x|w_1)}{p(x|w_2)} = \frac{P(w_2)}{P(w_1)} \cdot \frac{\lambda_{21}}{\lambda_{12}} \qquad (VIII)}$$

#6

12: Letting $P(\omega_1) = K \cdot P(\omega_2)$ where $K \geq 1$,

how can we modify the penalty terms $\{\lambda_{12}, \lambda_{21}\}$ so that the decision boundary does not move?

$x_0^*(1) = x_0^* = x_0^*(K)$, $K \geq 1$

(i): <u>When $K = 1$:</u> $\frac{P(x_0^* | \omega_1)}{P(x_0^* | \omega_2)} = \frac{P(\omega_2)}{P(\omega_1)}$ {No Penalty Terms} $\Rightarrow$

$$\boxed{\frac{P(x_0^* | \omega_1)}{P(x_0^* | \omega_2)} = 1}$$

(ii): When $K > 1$: $\frac{P(x_0^*(K) | \omega_1)}{P(x_0^*(K) | \omega_2)} = \frac{P(\omega_2)}{K \cdot P(\omega_2)} \cdot \frac{\lambda_{21}}{\lambda_{12}}$ {Penalty Terms} $\Rightarrow$

$$\frac{P(x_0^*(K) | \omega_1)}{P(x_0^*(K) | \omega_2)} = \frac{1}{K} \cdot \frac{\lambda_{21}}{\lambda_{12}} = 1 \Rightarrow \frac{\lambda_{21}}{\lambda_{12}} = K \Rightarrow \boxed{\lambda_{21} = K \cdot \lambda_{12}}$$