Moment Generating Functions

- In probability and statistics, the moment generating function (mgf) (ΡΟΠΟΓΕΝΝΗΤΡΙΑ ΣΥΝΑΡΤΗΣΗ) of a random variable constitutes an alternative way to specify its probability distribution.

- Required Knowledge: CUMULATIVE DISTRIBUTION FUNCTION) (Αθροιστική Συνάρτηση Κατανομής) [CDF]

• The cumulative distribution function of a real-valued random variable X is given by:

$$\boxed{F_X(x) = P(X \leq x)} \quad (1)$$

• In this setting, the probability density function of a continuous random variable can be determined from the cumulative distribution function as:

$$\boxed{f_X(x) = \frac{dF_X(x)}{dx}} \quad (2)$$

• Likewise, the CDF of a continuous random variable X can be expressed as the integral of its pdf as:

$$\boxed{F_X(x) = \int_{-\infty}^{x} f_X(x)\,dx} \quad (3)$$

(*) S.O.S: In case of a random variable X which has a discrete component at a value b, we can write that:

$$P(X=b) = F_X(b) - \lim_{x \to b^-} F_X(x)$$

If $F_X(\cdot)$ is continuous at b, $P(X=b) = 0$ and there is no discrete component at b.

- **MGF DEFINITION:** Let X be a random variable with CDF $F_X(\cdot)$. The moment generating function (mgf) of X (or F_X), denoted by $M_X(t)$ is defined as:

$$\boxed{M_X(t) = E[e^{tX}]}$$

given that the expectation exists for t in some neighborhood of $\emptyset$.

- The moment generating function is so called because it can be used to find the moments of the underlying distribution. Considering the Taylor series expansion of e^{tX} we get:

$$\boxed{e^{tX} = 1 + tX + \frac{t^2}{2!}X^2 + \frac{t^3}{3!}X^3 + \dots + \frac{t^n}{n!}X^n + \dots\dots}$$

- The term $\boxed{m_n = E[X^n]}$ identifies the n-th moment of X.

- Hence, we can write that:

$$\boxed{M_X(t) = E[e^{tX}] = 1 + tE[X] + \frac{t^2}{2!}E[X^2] + \frac{t^3}{3!}E[X^3] + \dots \frac{t^n}{n!}E[X^n] + \dots}$$

- The last formula suggests that we may express the MGF $M_X(t)$ of a random variable X in the following way:

$$\boxed{M_X(t) = \sum_{n=0}^{\infty} \frac{t^n}{n!} \cdot m_n} \quad \text{where} \quad \boxed{m_n = E[X^n]}$$

(n-th order moment)

- The previous formulation suggests that if we are in position of acquiring the power series expansion of $M_X(t)$ in the following way:

$$\boxed{M_X(t) = \sum_{n=0}^{\infty} a_n t^n}$$

then by equating the respective terms of the two power series we can write that:

$$\boxed{m_n = n!\, a_n}$$

- **Theorem**: If X has mgf $M_X(t)$, then

$$\boxed{E[X^n] = M_X^{(n)}(0)}$$

where we define that:

$$\boxed{M_X^{(n)}(0) = \frac{d^n}{dt^n} M_X(t) \Big|_{t=0}}$$

This entails that the n-th order moment of X is equal to the n-th derivative of $M_X(t)$ evaluated at $t=0$.

Example: Compute the mgf of a standardized normally distributed random variable $X \sim N(0,1)$.

Solution: We know that the pdf of X is given by:

$$\boxed{f_X(x) = \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{1}{2}x^2} \quad (1) \quad , \quad -\infty < x < +\infty}$$

The mgf of the random variable X will be computed as:

$$\boxed{M_X(t) = E\left[e^{tX}\right] = \int_{-\infty}^{+\infty} e^{tx} \cdot \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{1}{2}x^2} dx \quad (2)}$$

This integral looks rather complicated to be evaluated, but it may be computed by utilizing the following trick:

(*) Collect the exponential terms and complete the square:

$$\boxed{e^{tx} \cdot e^{-\frac{1}{2}x^2} = e^{u} \quad (3)} \quad \text{where} \quad \boxed{u = -\frac{x^2}{2} + tx \quad (4)}$$

▸ At this point, we would like to express quantity u in the following form:

$$\boxed{u = \lambda(x + at)^2 + \mu = \lambda x^2 + 2\lambda x a t + a^2 t^2 \lambda + \mu = -\frac{x^2}{2} + tx \quad (5)}$$

▸ EQUATE POLYNOMIAL COEFFICIENTS:

$$\begin{cases} \lambda = -\frac{1}{2} \\ 2a\lambda = 1 \\ \mu + a^2 t^2 \lambda = 0 \end{cases} \Rightarrow \begin{cases} \boxed{\lambda = -\frac{1}{2} \quad (6)} \\ \boxed{a = \frac{1}{2\lambda} = -1 \quad (7)} \\ \boxed{\mu = -a^2 t^2 \lambda = +\frac{1}{2}t^2 \quad (8)} \end{cases} \Rightarrow \boxed{u = -\frac{1}{2}(x-t)^2 + \frac{1}{2}t^2 \quad (9)}$$

• Taking into consideration the previous derivations, we may write that:

$$M_X(t) = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{1}{2}(x-t)^2 + \frac{1}{2}t^2} \, dx \Rightarrow$$

$$M_X(t) = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{1}{2}(x-t)^2} \cdot e^{\frac{t^2}{2}} \, dx \Rightarrow$$

$$\boxed{M_X(t) = e^{\frac{t^2}{2}} \cdot \underbrace{\boxed{\int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{(x-t)^2}{2}} \, dt}}_{I}} \quad (10)$$

• The integral I appearing in Eq. (10) corresponds to the integration of the probability density function of a random variable $x' \sim N(t, 1)$ and is by definition equal to 1. ($I = 1$).

• Therefore, we have that:
$$\boxed{M_X(t) = e^{\frac{t^2}{2}}} \quad (11)$$

(*) In this context, the power series expansion of $M_X(t)$ yields:

$$M_X(t) = e^{\frac{t^2}{2}} = \sum_{n=0}^{\infty} \frac{1}{n!} \cdot \left(\frac{t^2}{2}\right)^n = \sum_{n=0}^{\infty} \frac{t^{2n}}{n!\,2^n} \Rightarrow$$

$$\boxed{M_X(t) = \sum_{n=0}^{\infty} \frac{t^{2n}}{n!\,2^n}} \quad (12)$$

(*) We have also provided the expression: $\boxed{M_x(t) = \sum_{n=0}^{\infty} \frac{t^n}{n!} m_n}$ (13)

(*) Expression (13) can be decomposed into odd and even terms as:

$$\boxed{M_x(t) = \sum_{n=0}^{\infty} \frac{t^{2n}}{(2n)!} \cdot m_{2n} + \sum_{n=0}^{\infty} \frac{t^{2n+1}}{(2n+1)!} \cdot m_{2n+1}} \quad (14)$$

(*) Eq. (14) can take the form of Eq. (12) by zeroing out all odd terms such that $\boxed{m_{2n+1} = 0, \ \forall n \in \mathbb{N}}$

(*) Subsequently, be equating the respective terms of the two power series, we get:

$$\frac{m_{2n}}{(2n)!} = \frac{1}{n!\,2^n} \Longrightarrow \boxed{m_{2n} = \frac{(2n)!}{n!\,2^n}} \quad (15)$$

(*) Finally, we can write that:

$$\boxed{\begin{cases} m_{2n+1} = \infty, \ \forall n \in \mathbb{N} \\ m_{2n} = \dfrac{(2n)!}{n!\,2^n}, \ \forall n \in \mathbb{N} \end{cases}} \quad (15)$$

$$M_x(t) = e^{\frac{t^2}{2}}$$

$$\boxed{\begin{aligned} M_x^{(1)}(t) &= t \cdot M_x(t) \\ M_x^{(2)}(t) &= (t^2+1) \cdot M_x(t) \\ M_x^{(3)}(t) &= (t^3+3t) M_x(t) \\ M_x^{(4)}(t) &= (t^4+6t^2+3) M_x(0) \end{aligned}}$$